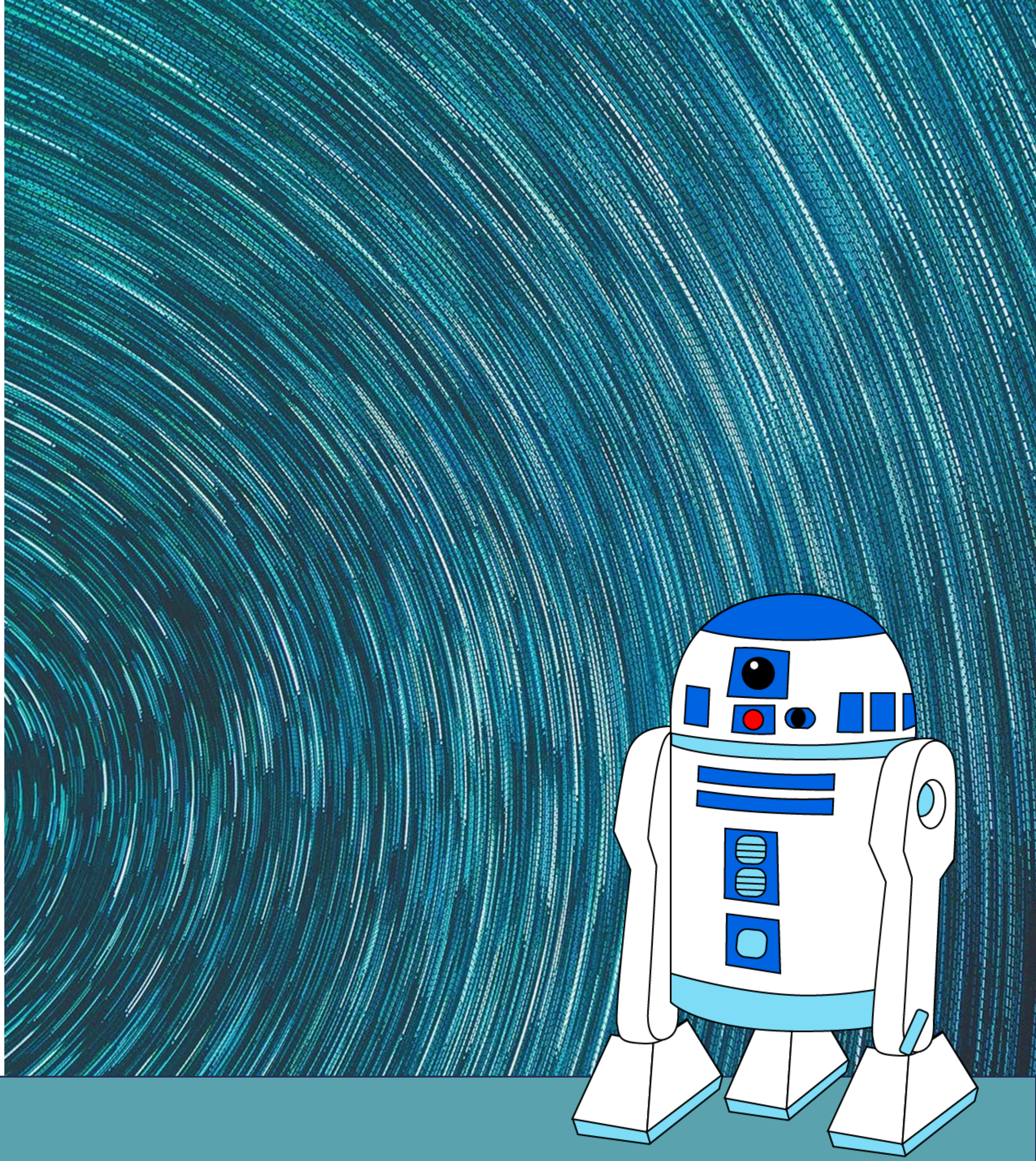
CIS 421/521:
ARTIFICIAL INTELLIGENCE

Logistic Regression

Jurafsky and Martin Chapter 5



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Classifier components

- Machine learning classifiers require a training corpus of M observations input/output pairs $(x^{(i)}, y^{(i)})$.

1. A **feature representation** of the input. For each input observation $x^{(i)}$, this will be a vector of features $[x_1, x_2, ..., x_n]$.
2. A **classification function** that computes the estimated class $\hat{y}$ via $p(y|x)$.
3. An **objective function** for learning, usually involving minimizing error on training examples.
4. An algorithm for **optimizing** the objective function.

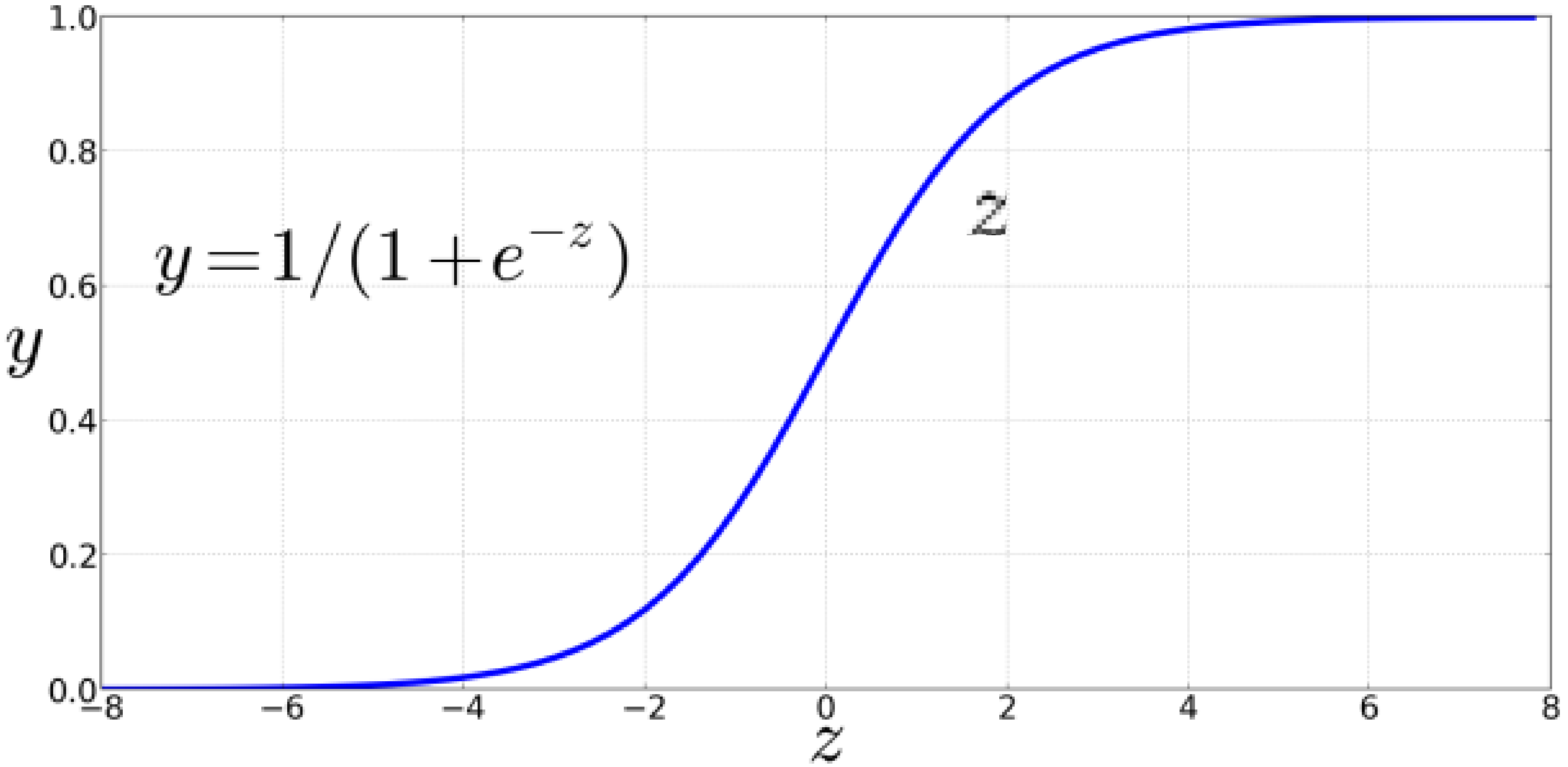
Sentiment classifier

- **Input:** "Spiraling away from narrative control as its first three episodes unreel, this series, about a post-apocalyptic future in which nearly everyone is blind, wastes the time of Jason Momoa and Alfre Woodard, among others, on a story that starts from a position of fun, giddy strangeness and drags itself forward at a lugubrious pace."
- **Output: positive (1)** or **negative (0)**

Sentiment classifier

- For sentiment classification, consider an input observation *x*, represented by a vector of **features** $[x_1, x_2, \ldots, x_n]$. The classifier output *y* can be 1 (positive sentiment) or 0 (negative sentiment). We want to estimate ***P*(*y* = 1|*x*)**.
- Logistic regression solves this task by learning, from a training set, a vector of **weights** and a **bias term**.
- $z = \sum_i w_i x_i + b$
- We can also write this as a dot product:
- $z = w \cdot x + b$

Sigmoid function



Probabilities

$$P(y=1)=\sigma(w\cdot x+b)=\frac{1}{1+e^{-(w\cdot x+b)}}$$

Decision boundary

- Now we have an algorithm that given an instance *x* computes the probability $P(y = 1|x)$. How do we make a decision?

$$\hat{y} = \begin{cases} 1\ if\ P(y = 1|x) > 0.5 \\ 0\ otherwise \end{cases}$$

- For a test instance *x*, we say **yes** if the probability $P(y = 1|x)$ is more than .5, and **no** otherwise. We call .5 the decision boundary

Extracting Features

- It's hokey. There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

Var	Definition	Value
x_1	Count of positive lexicon words	
x_2	Count of negative lexicon words	
x_3	Does no appear? (binary feature)	
x_4	Number of 1st and 2nd person pronouns	
x_5	Does ! appear? (binary feature)	
x_6	Log of the word count for the document	

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Var	Definition	Value
x_1	Count of positive lexicon words	3
x_2	Count of negative lexicon words	
x_3	Does no appear? (binary feature)	
x_4	Number of 1st and 2nd person pronouns	
x_5	Does ! appear? (binary feature)	
x_6	Log of the word count for the document	

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Var	Definition	Value
x_1	Count of positive lexicon words	3
x_2	Count of negative lexicon words	2
x_3	Does no appear? (binary feature)	
x_4	Number of 1st and 2nd person pronouns	
x_5	Does ! appear? (binary feature)	
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Var	Definition	Value
x_1	Count of positive lexicon words	3
x_2	Count of negative lexicon words	2
x_3	Does no appear? (binary feature)	1
x_4	Number of 1st and 2nd person pronouns	
x_5	Does ! appear? (binary feature)	
x_6	Log of the word count for the document	

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x_3	Does no appear? (binary feature)	1
x_4	Number of 1^{st} and 2nd person pronouns	3
x_5	Does ! appear? (binary feature)	0
x_6	Log of the word count for the document	

Extracting Features

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Word count = 64, ln(64) = 4.15

Var	Definition	Value
x_1	Count of positive lexicon words	3
x_2	Count of negative lexicon words	2
x_3	Does no appear? (binary feature)	1
x_4	Number of 1st and 2nd person pronouns	3
x_5	Does ! appear? (binary feature)	0
x_6	Log of the word count for the document	4.15

Var	Definition	Value	Weight	Product
x_1	Count of positive lexicon words	3	2.5	
x_2	Count of negative lexicon words	2	-5.0	
x_3	Does no appear? (binary feature)	1	-1.2	
x_4	Num 1st and 2nd person pronouns	3	0.5	
x_5	Does ! appear? (binary feature)	0	2.0	
x_6	Log of the word count for the doc	4.15	0.7	
b	bias	1	0.1	

$$z = \sum_i w_i x_i + b$$

Computing Z

Var	Definition	Value	Weight	Product
x_1	Count of positive lexicon words	3	2.5	7.5
x_2	Count of negative lexicon words	2	-5.0	-10
x_3	Does no appear? (binary feature)	1	-1.2	-1.2
x_4	Num 1st and 2nd person pronouns	3	0.5	1.5
x_5	Does ! appear? (binary feature)	0	2.0	0
x_6	Log of the word count for the doc	4.1	0.7	2.905
b	bias	1	0.1	0.1

$$z = \sum_i w_i x_i + b$$

z=0.805

Sigmoid(Z)

Var	Definition	Value	Weight	Product
x_1	Count of positive lexicon words	3	2.5	7.5
x_2	Count of negative lexicon words	2	-5.0	-10
x_3	Does no appear? (binary feature)	1	-1.2	-1.2
x_4	Num 1^{st} and 2nd person pronouns	3	0.5	1.5
x_5	Does ! appear? (binary feature)	0	2.0	0
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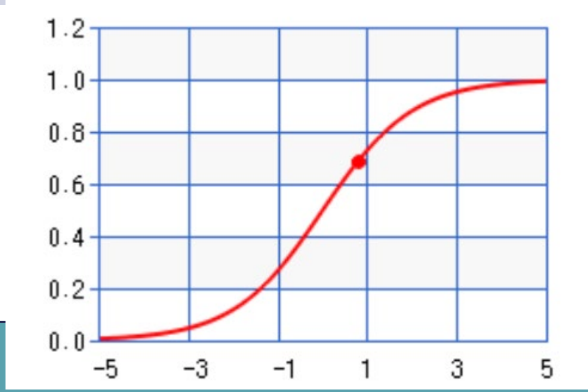

$$\sigma(0.805)=0.69$$

Learning in logistic regression

- How do we get the weights of the model? We learn the parameters (weights + bias) via learning. This requires 2 components:

1. An objective function or **loss function** that tells us *distance* between the system output and the gold output. We will use **cross-entropy loss**.
2. An algorithm for optimizing the objective function. We will use stochastic gradient descent to **minimize** the **loss function.**

Loss functions

- We need to determine for some observation *x* how close the classifier output ($\hat{y}$= σ (*w* · *x* + *b*)) is to the correct output (*y*, which is 0 or 1).
  - $L(\hat{y}, y) =$ how much $\hat{y}$ differs from the true *y*
- One example is mean squared error
  - $L_{MSE}(\hat{y}, y) = \frac{1}{2}(\hat{y} - y)^2$

Loss functions for probabilistic classification

- We use a loss function that prefers the correct class labels of the training example to be more likely.
- Conditional maximum likelihood estimation: Choose parameters *w, b* that maximize the (log) probabilities of the true labels in the training data.
- The resulting loss function is the negative log likelihood loss, more commonly called the **cross entropy loss.**

Loss functions for probabilistic classification

- For one observation x, let's **maximize** the probability of the correct label p(y|x).
  - $p(y|x) = \hat{y}^y(1-\hat{y})^{1-y}$
- If y = 1, then $\mathrm{p(y|x)} = \hat{y}$.
- If y = 0, then $\mathrm{p(y|x)} = 1 - \hat{y}$.

Loss functions for probabilistic classification

- Change to logs (still maximizing)
  - $\log p(y|x) = \log[\hat{y}^y (1-\hat{y})^{1-y}]$
  - $= y \log \hat{y} + (1-y) \log(1-\hat{y})$
- This tells us what log likelihood should be maximized. But for loss functions, we want to minimize things, so we'll flip the sign.

Cross-entropy loss

- **The result is cross-entropy loss:**
- $L_{CE}(\hat{y}, y) = -\log p(y|x) = -[y \log \hat{y} + (1 - y) \log(1 - \hat{y})]$

- **Finally, plug in the definition for** $\hat{\boldsymbol{y}} = \sigma(w \cdot x) + b$
- $L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log(1 - \sigma(w \cdot x + b))]$

Cross-entropy loss

- Why does minimizing this negative log probability do what we want? We want the **loss** to be **smaller** if the model's estimate is **close to correct**, and we want the **loss** to be **bigger if it is confused.**

It's hokey. There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

P(sentiment=1|It's hokey...) = 0.69. Let's say y=1.

$$L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x+b) + (1-y)\log(1-\sigma(w \cdot x+b))]$$

$$= -[\log \sigma(w \cdot x+b)]$$

$$= -\log(0.69) = \mathbf{0.37}$$

Cross-entropy loss

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P(sentiment=1|It's hokey...) = 0.69. Let's **pretend** y=0.

$$L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x+b) + (1-y)\log(1-\sigma(w \cdot x+b))]$$

$$= -[\log(1-\sigma(w \cdot x+b))]$$

$$= -\log(0.31) = \mathbf{1.17}$$

Cross-entropy loss

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If our prediction is **correct**, then our CE loss is **lower**

$$= -\log(0.69) = \mathbf{0.37}$$

If our prediction is **incorrect**, then our CE loss is **higher**

$$-\log(0.31) = \mathbf{1.17}$$

Loss on all training examples

$$\log p(training\ labels) = \log \prod_{i=1}^{m} p(y^{(i)}|x^{(i)})$$

$$= \sum_{i=1}^{m} \log p(y^{(i)}|x^{(i)})$$

$$= -\sum_{i=1}^{m} \mathrm{L}_{\mathrm{CE}}(\hat{y}^{(i)}|y^{(i)})$$

Finding good parameters

- We use gradient descent to find good settings for our weights and bias by minimizing the loss function.

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \frac{1}{m} \sum_{i=1}^{m} L_{CE}(y^{(i)}, x^{(i)}; \theta)$$

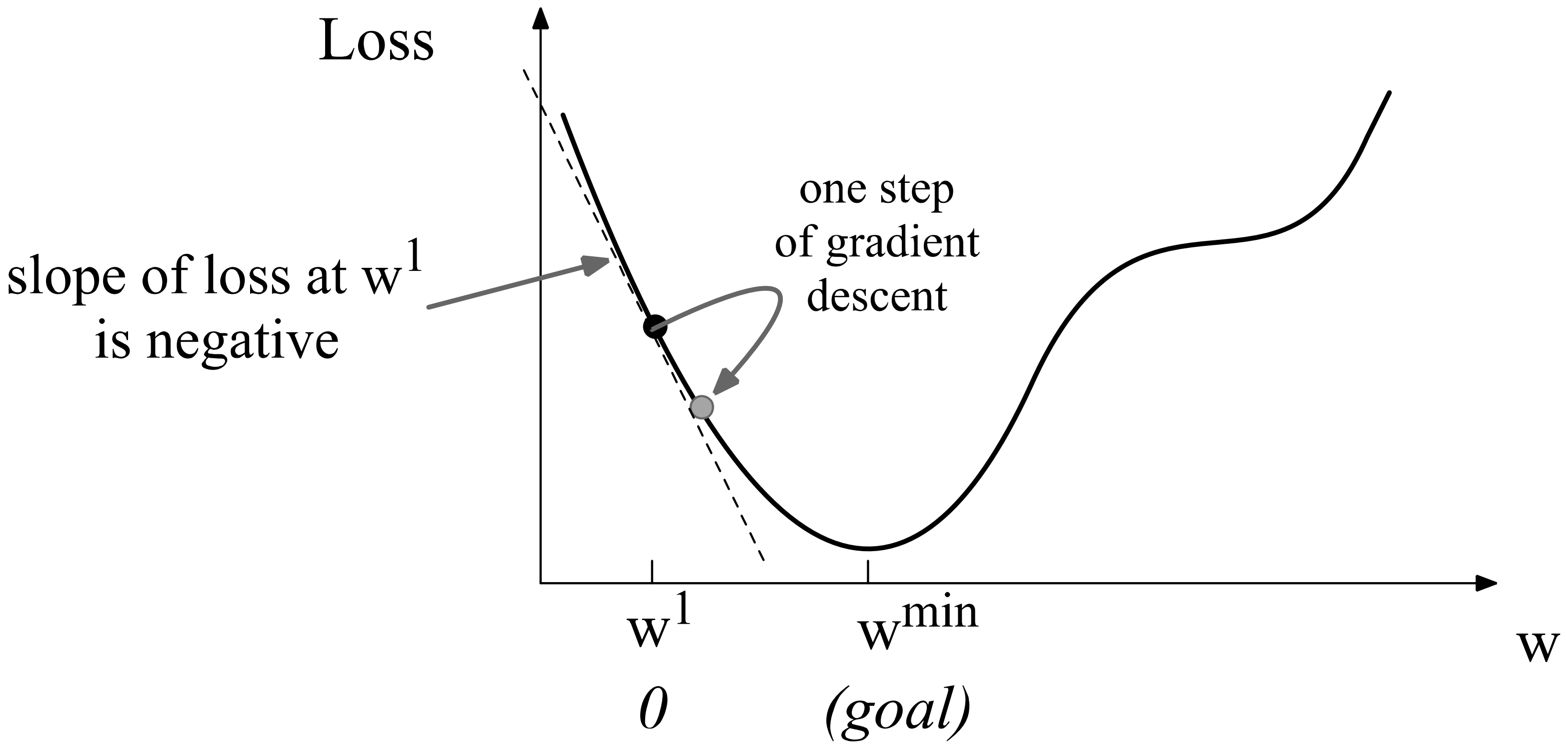
- **Gradient descent** is a method that finds a minimum of a function by figuring out in which direction (in the space of the parameters θ) the function's slope is rising the most steeply, and moving in the opposite direction.

Gradient descent

Global v. Local Minimums

- For logistic regression, this loss function is conveniently **convex**.
- A convex function has just **one minimum**, so there are no local minima to get stuck in.
- So gradient descent starting from any point is guaranteed to find the minimum.

Iteratively find minimum

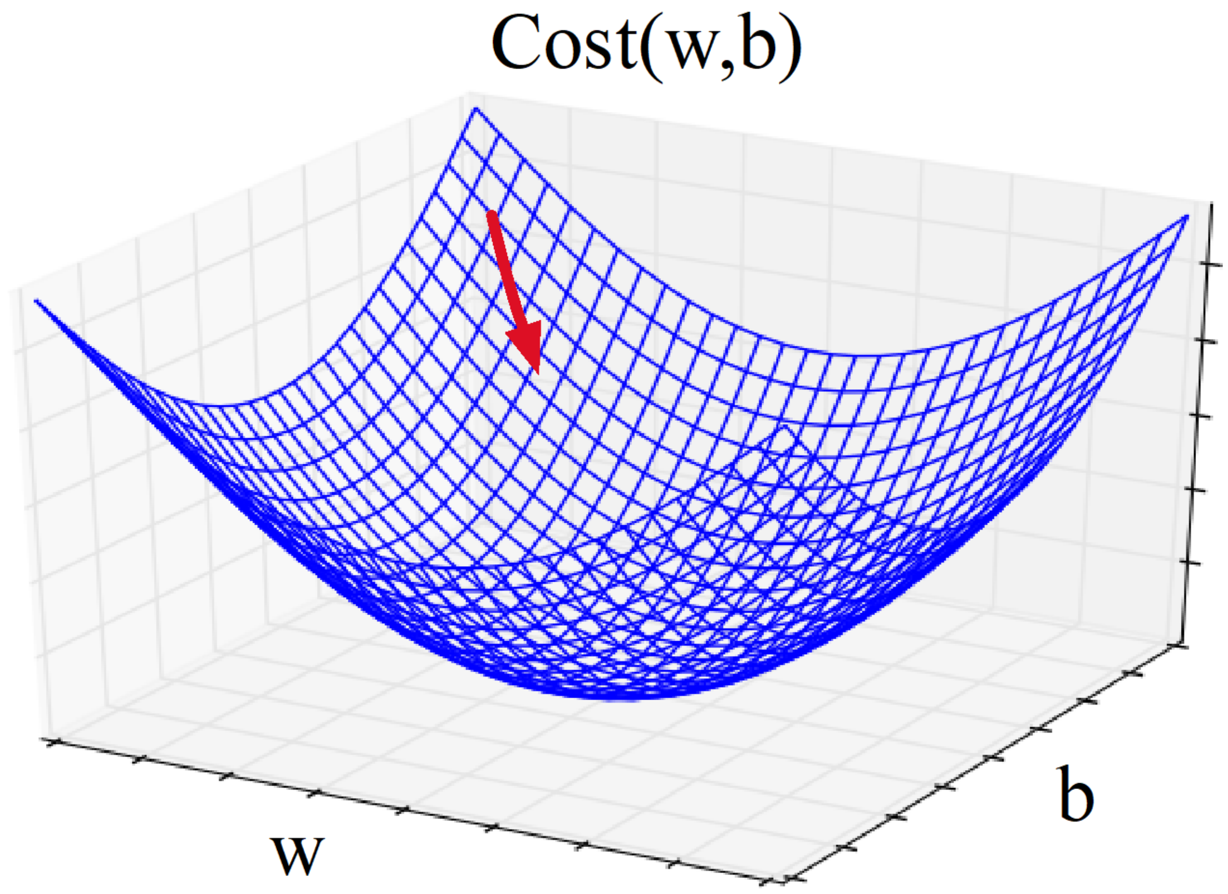


How much should we update the parameter by?

- The magnitude of the amount to move in gradient descent is the value of the slope weighted by a learning rate η.
- A higher/faster learning rate means that we should move *w* more on each step.

$$w^{t+1} = w^t - \eta \frac{d}{dw} f(x; w)$$

Many dimensions



Stochastic gradient descent algorithm

function STOCHASTIC GRADIENT DESCENT($L(), f(), x, y$) **returns** θ

where: L is the loss function

f is a function parameterized by θ

x is the set of training inputs $x^{(1)},\ x^{(2)}, \ldots,\ x^{(n)}$

y is the set of training outputs (labels) $y^{(1)},\ y^{(2)}, \ldots,\ y^{(n)}$

$\theta \leftarrow 0$

repeat T times

For each training tuple $(x^{(i)},\ y^{(i)})$ (in random order)

Compute $\hat{y}^{(i)} = f(x^{(i)};\theta)$ # What is our estimated output $\hat{y}$?

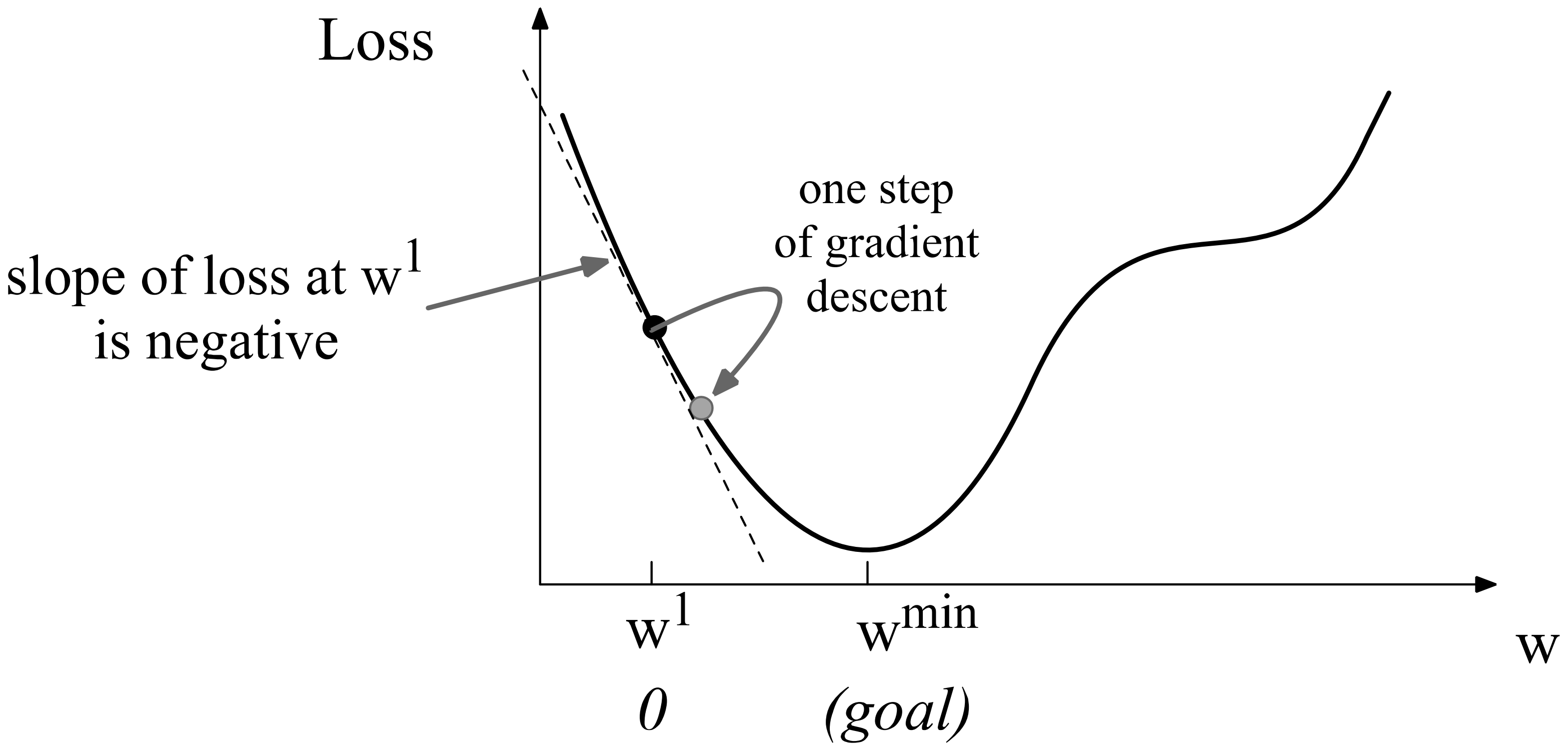
Compute the loss $L(\hat{y}^{(i)}, y^{(i)})$ # How far off is $\hat{y}^{(i)})$ from the true output $y^{(i)}$?

$g \leftarrow \nabla_\theta L(f(x^{(i)};\theta), y^{(i)})$ # How should we move θ to maximize loss ?

$\theta \leftarrow \theta - \eta\ g$ # go the other way instead

return θ

Iteratively find minimum



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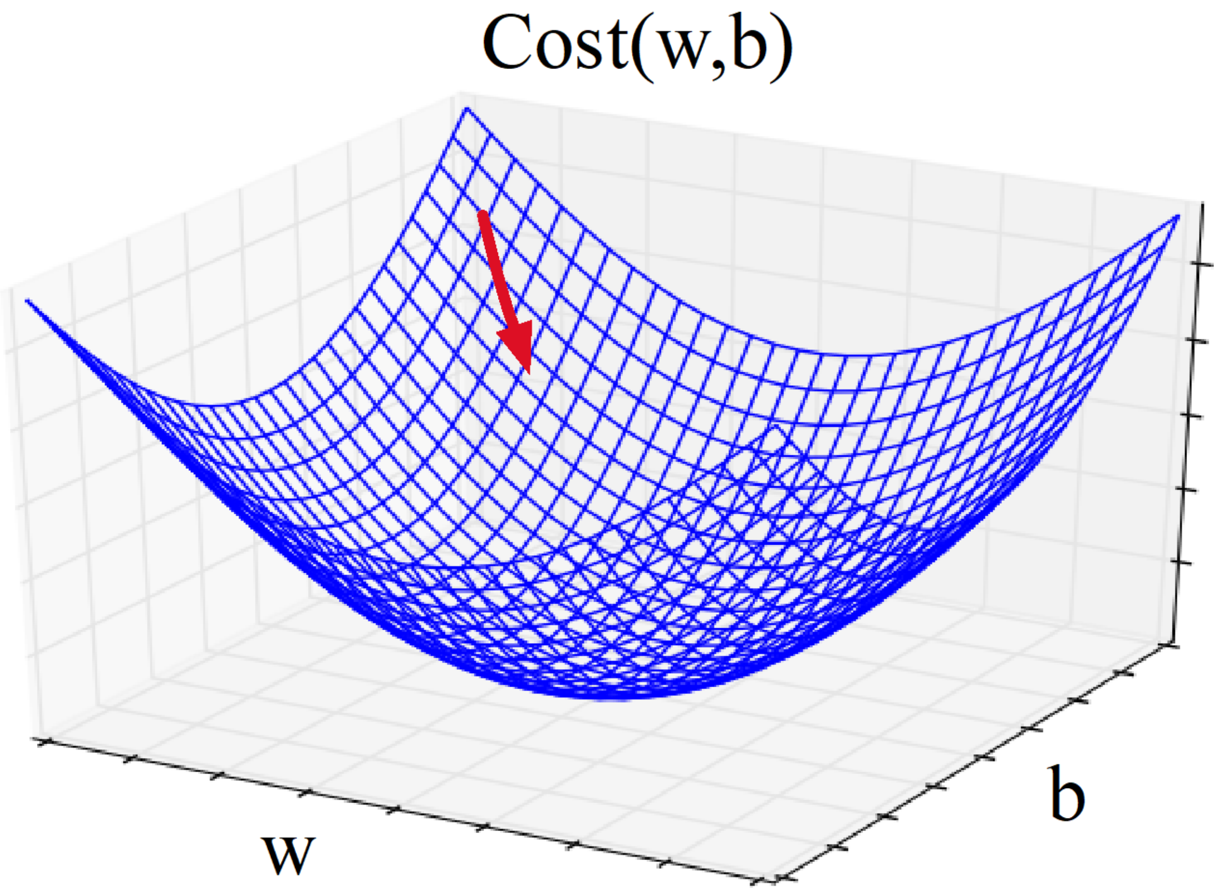

Updating each dimension w_i

$$\nabla_\theta L(f(x;\theta),y)) = \begin{bmatrix} \frac{\partial}{\partial w_1} L(f(x;0),y) \\ \frac{\partial}{\partial w_2} L(f(x;0),y) \\ \vdots \\ \frac{\partial}{\partial w_n} L(f(x;0),y) \end{bmatrix}$$

The final equation for updating θ based on the gradient is $\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta),y)$

The Gradient

- To update θ, we need a definition for the gradient $\nabla L(f(x;\theta), y)$.
- For logistic regression, the cross-entropy loss function is:

$$L_{CE}(w,b) = -[y \log \sigma(w \cdot x + b) + (1-y)\log(1-\sigma(w \cdot x + b))]$$

- The derivative of this function for one observation vector *x* for a single weight w_j is

$$\frac{\partial L_{CE}(w,b))}{\partial w_j} = [\sigma(w \cdot x + b) - y]x_j$$

- The gradient is a very intuitive value: the difference between the true *y* and our estimate for x, multiplied by the corresponding input value x_j .

Average Loss

$$Cost(w,b) = \frac{1}{m}\sum_{i=1}^{m} L_{CE}(\hat{y}^{(i)}, y^{(i)})$$

$$= -\frac{1}{m}\sum_{i=1}^{m} y^{(i)} \log \sigma\left(w \cdot x^{(i)} + b\right) + \left(1 - y^{(i)}\right)\log(1 - \sigma\left(w \cdot x^{(i)} + b\right))$$

This is what we want to minimize!!

The Gradient

- The loss for a batch of data or an entire dataset is just the average loss over the *m* examples

$$Cost(w,b) = -\frac{1}{m}\sum_{i=1}^{m} y^{(i)} \log \sigma\left(w \cdot x^{(i)} + b\right) + \left(1 - y^{(i)}\right) \log(1 - \sigma\left(w \cdot x^{(i)} + b\right))$$

- The gradient for multiple data points is the sum of the individual gradients:

$$\frac{\partial Cost(w,b)}{\partial w_j} = \sum_{i=1}^{m} [\sigma\left(w \cdot x^{(i)} + b\right) - y^{(i)}] x_j^{(i)}$$

Worked example

- Let's walk though a single step of the gradient descent algorithm. We'll use a simple sentiment classifier with just 2 features, and 1 training instance where the correct value is *y* = 1 (this is a positive review).
  - $x_1 = 3$ (count of positive lexicon words)
  - $x_2 = 2$ (count of positive negative words)
- The initial weights and bias in θ^0 are all set to 0, and the initial learning rate η is 0.1:
  - $w_1 = w_2 = b = 0$
  - $\eta = 0.1$
- The single update step requires that we compute the gradient, multiplied by the learning rate:

$$\theta^{t+1} = \theta^t - \eta \nabla_\theta L(f(x^{(i)}; \theta), y^{(i)})$$

Worked example

- The derivative of this function for **a single training example** *x* for a single weight w_j is

$$\frac{\partial L_{CE}(w,b)}{\partial w_j} = [s(w \cdot x + b) - y]x_j$$

- The gradient vector has 3 dimensions, for w_1, w_2, and b.
  For our input, x_1 = 3 and x_2 = 2
  - x_2 = 2

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{CE}(w,b)}{\partial w_1} \\ \frac{\partial L_{CE}(w,b)}{\partial w_2} \\ \frac{\partial L_{CE}(w,b)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot\ x + b) - y)x_1 \\ (\sigma(w \cdot\ x + b) - y)x_2 \\ \sigma(w \cdot\ x + b) - y \end{bmatrix} = \begin{bmatrix} (\sigma(0) - 1)x_1 \\ (\sigma(0) - 1)x_2 \\ \sigma(0) - 1 \end{bmatrix} = \begin{bmatrix} -0.5x_1 \\ -0.5x_2 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix}$$

Worked example

- Now that we have a gradient $\nabla_{w,b}$, we compute the new parameter vector θ^1 by moving θ^0 in the opposite direction from the gradient:

$$\theta^1 = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} - \eta \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} .15 \\ .1 \\ .05 \end{bmatrix}$$

- So after one step of gradient descent, the weights have shifted to be:
  - w_1 = 0.15, w_2 = 0.1, and b= .05

Mini-batch training

- **Stochastic** gradient descent chooses a **single random example** at a time and updates its weights on that example. As a result the updates can fluctuate.
- An alternate is **batch training,** which computes the gradient over the **entire dataset**. This gives a much better estimate of which direction to move the weights but takes a long time to compute.
- A commonly used compromise is **mini-batch training**, where we train on a small batch. The batch size can be 512 or 1024, often selected based on computational resources, so that all examples in the mini-batch can be processed in parallel. The loss is then accumulated.

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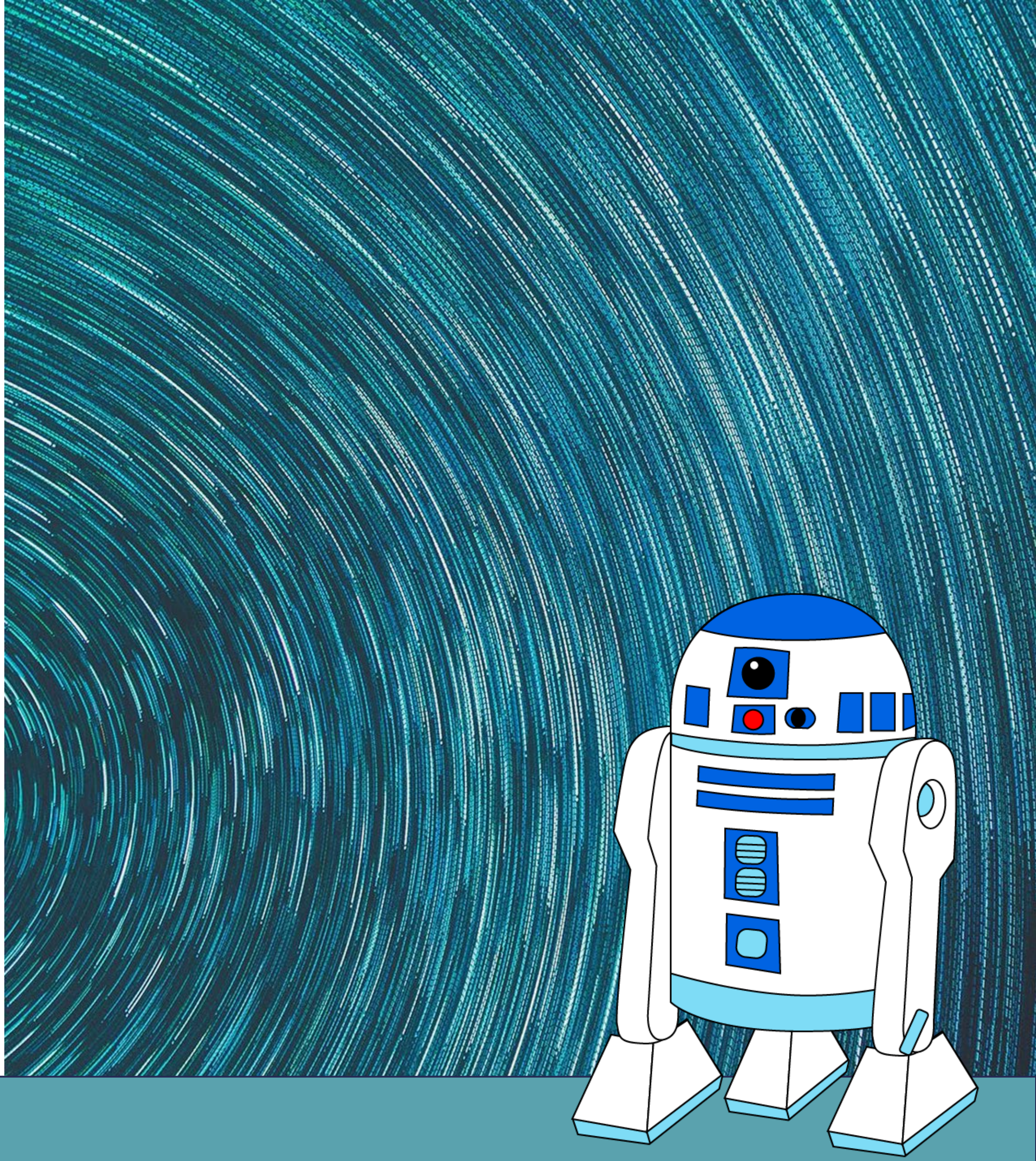
Logistic Regression – Wrap Up

Jurafsky and Martin Chapter 5



Regularization

- **Overfitting** is a problem with many machine learning models. Overfitting results in poor generalization and poor performance on unseen test set.
- In logistic regression, if a feature only occurs in one class then it will get a **high weight**. Sometimes we are just modelling noisy factors that just accidentally correlate with the class.
- **Regularization** is a way to penalize large weights. A regularization term is added to the loss function.
- Lasso regression uses L1 regularization
  Ridge regression uses L2 regularization

Multinomial logistic regression

- Instead of binary classification, we often want more than two classes. For sentiment classification we might extend the class labels to be **positive, negative**, and **neutral**.
- We want to know the probability of y for each class $c \in C$, $p(y = c|x)$.
- To get a proper probability, we will use a **generalization of the sigmoid function** called the **softmax function.**

$$\text{softmax}(z_i) = \frac{e^{z_j}}{\sum_{j=1}^{k} e^{z_j}} \; 1 \leq i \leq k$$

Softmax

- The softmax function takes in an input vector $z = [z_1, z_2, \ldots, z_k]$ and outputs a vector of values normalized into probabilities.

$$\mathrm{softmax}(z) = [\frac{e^{z_1}}{\sum_{i=1}^{k} e^{z_i}}, \frac{e^{z_2}}{\sum_{i=1}^{k} e^{z_i}}, \cdots, \frac{e^{z_k}}{\sum_{i=1}^{k} e^{z_i}}]$$

- For example, for this input:
  - z = [0.6, 1.1, −1.5, 1.2, 3.2, −1.1]
- Softmax will output:
  - [0.056, 0.090, 0.007, 0.099, 0.74, 0.010]