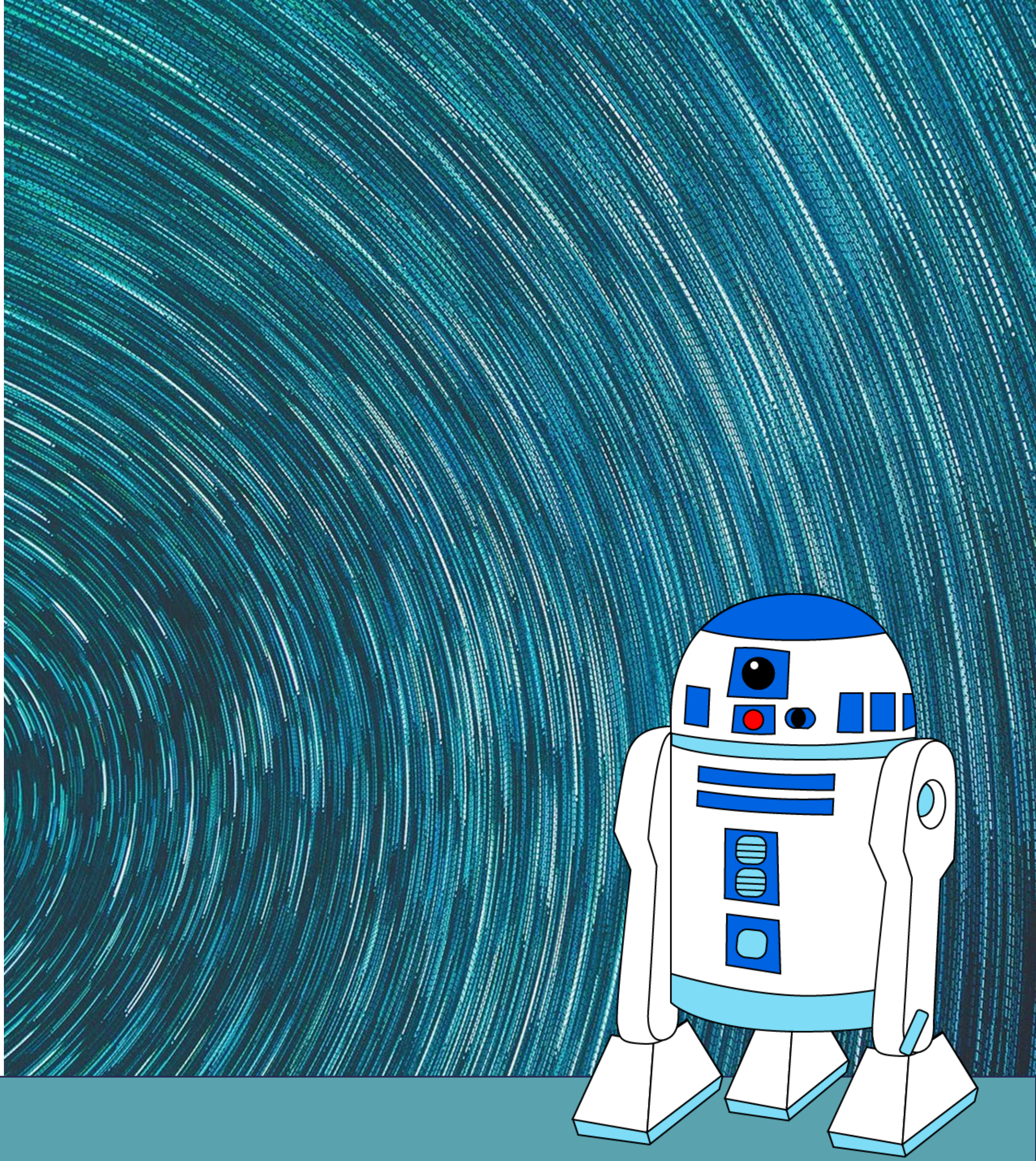
CIS 521:
ARTIFICIAL INTELLIGENCE

Constraint Satisfaction Problems

Harry Smith

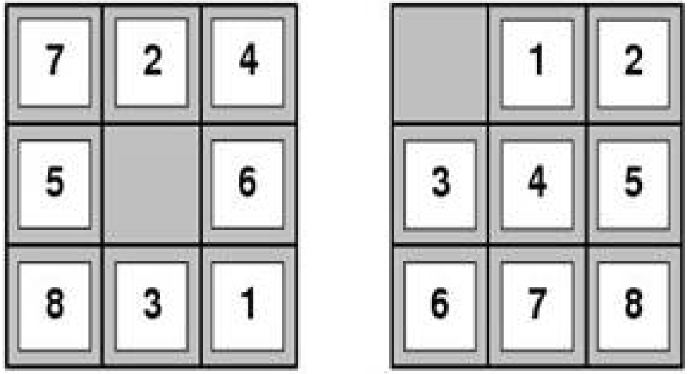


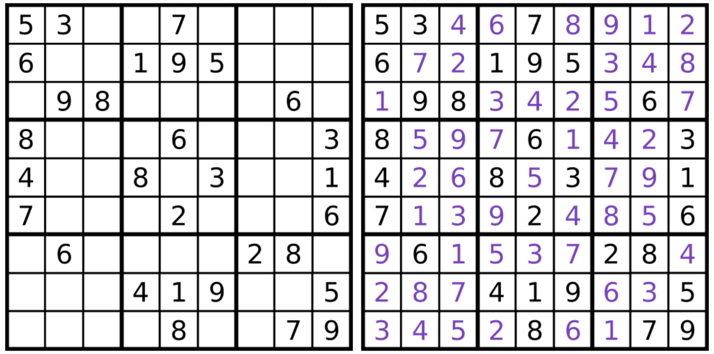
What is Search For?

- **Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space**

- **Planning: sequences of actions**
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- **Identification: assignments to variables**
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems





Big idea

- **Represent the *constraints* that solutions must satisfy in a uniform *declarative* language**
- **Find solutions by *GENERAL PURPOSE* search algorithms with no changes from problem to problem**
  - No hand-built transition functions
  - No hand-built heuristics
- **Just specify the problem in a formal declarative language, and a general-purpose algorithm does everything else!**

Constraint Satisfaction Problems

A CSP consists of:

- *Finite set of variables* $X_1, X_2, \ldots, X_n$
- *Nonempty* ***domain*** *of possible values* for each variable $D_1, D_2, \ldots D_n$ *where* $D_i = \{v_1, \ldots, v_k\}$
- *Finite set of constraints* $C_1, C_2, \ldots, C_m$
  - Each *constraint* C_i limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.

○ **A** *consistent* **assignment does not violate the constraints.**

○ **Example problem: Sudoku**

Constraints in Sudoku

8			4	7	3			
	2	6	8	5	1		9	
		5				8		
	1	3			8	4		
6		7	3		2	9		
	5			9	7	6		8
	6	2	7	3		5		
	3		2			7		6
4			6				2	

All different

Constraints in Sudoku

All different

8			4	7	3			
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		5				8		
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	3		2			7		6
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Constraints in Sudoku

All different

8			4	7	3			
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	5			9	7	6		8
	6	2	7	3		5		
	3		2			7		6
4			6				2	

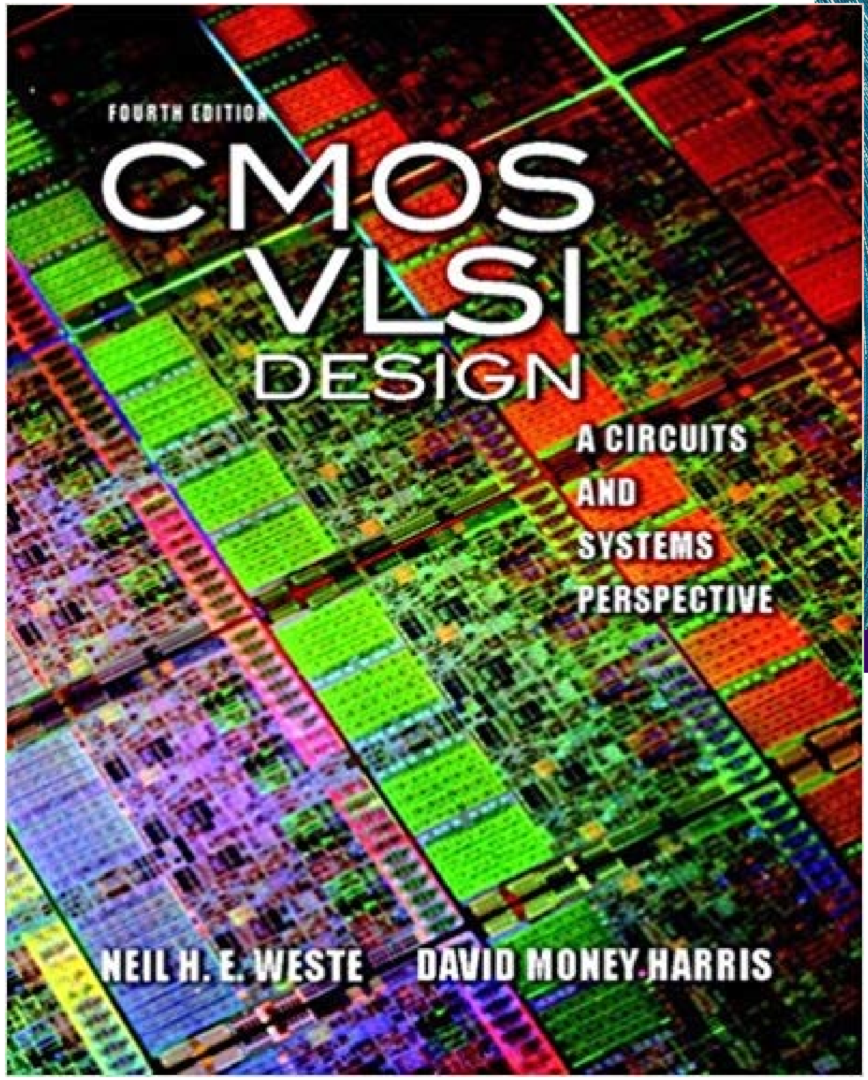
Constraint satisfaction problems

- An assignment is *complete* when every variable is assigned a value.
- A *solution* to a CSP is a *complete, consistent* assignment.
- Solutions to CSPs can be found by a completely *general purpose* algorithm, given only the formal specification of the CSP.

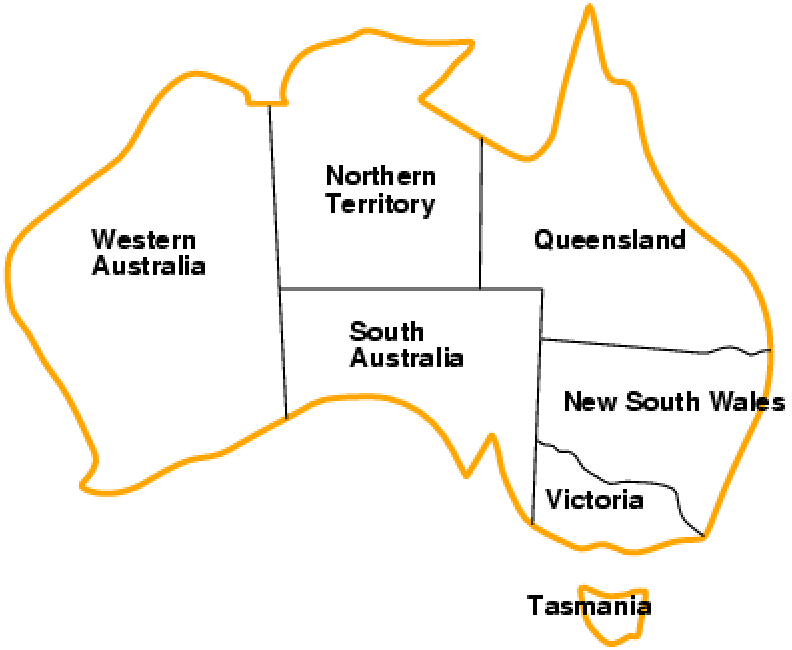
- Beyond our scope: CSPs that require a solution that maximizes an *objective function*.

Applications

- Map coloring
- Scheduling problems
  - Job shop scheduling
  - Scheduling the Hubble Space Telescope
- Floor planning for VLSI
- Sudoku
- …

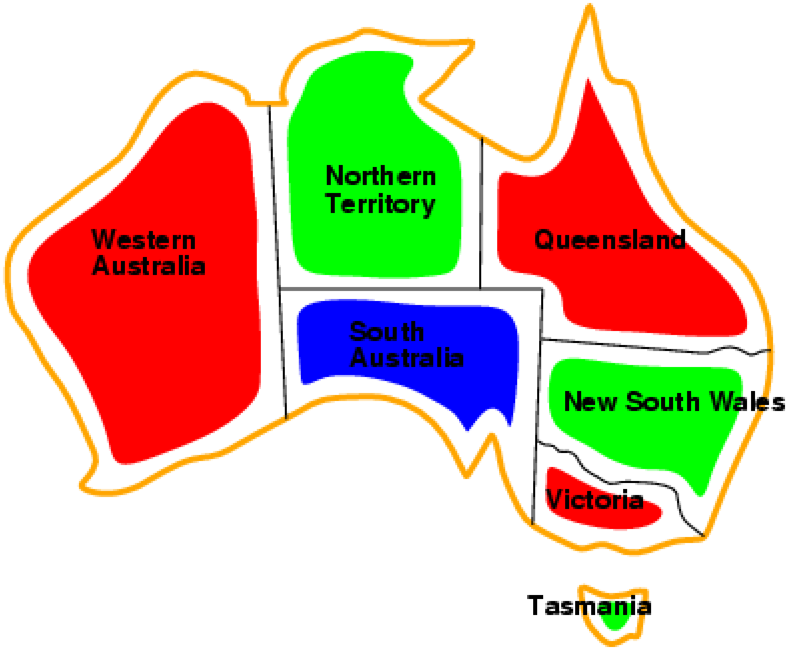


Example: Map-coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
  - e.g., WA ≠ NT
    - So (WA,NT) must be in {(red,green),(red,blue),(green,red), ...}

Example: Map-coloring



Solutions: complete and consistent assignments

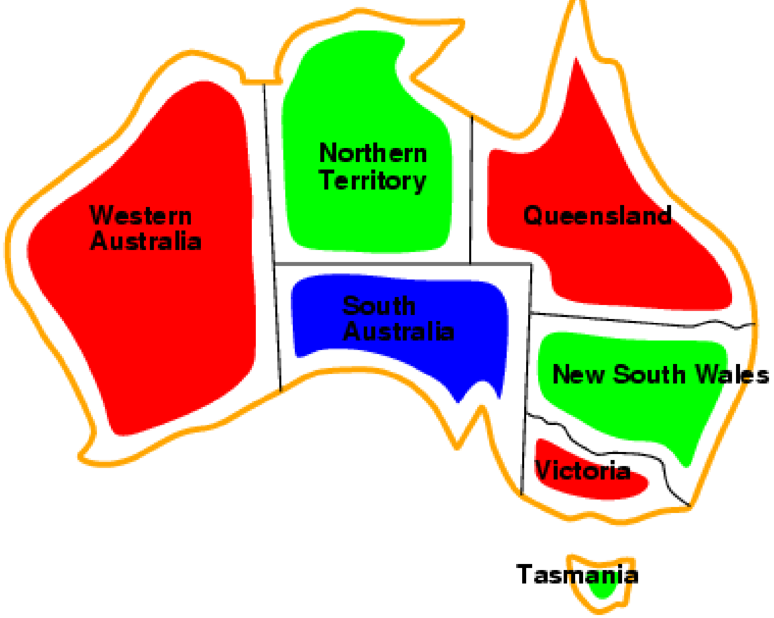
- e.g., WA = red, NT = green,Q = red, NSW = green, V = red, SA = blue, T = green

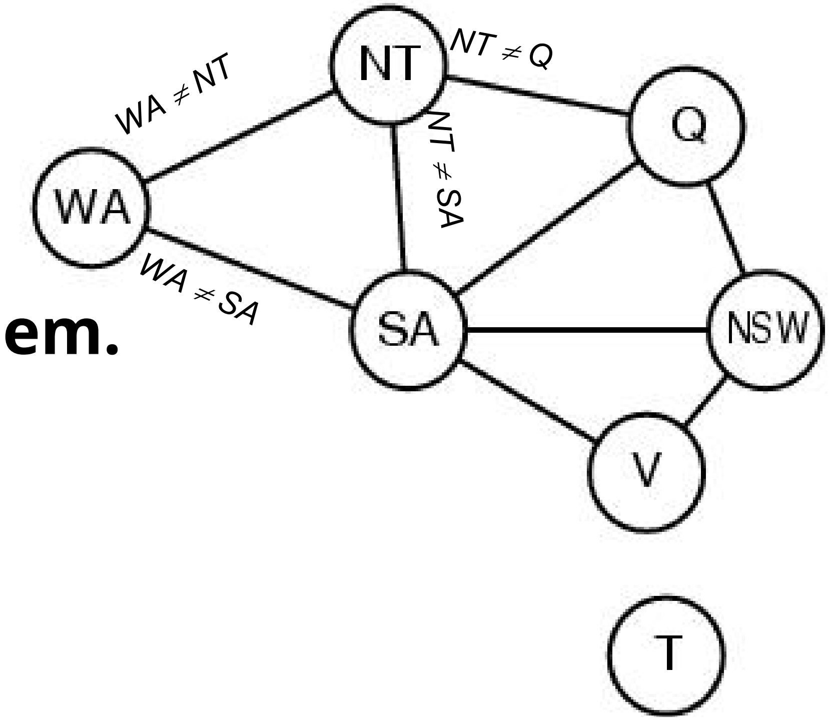
Benefits of CSP

- Clean specification of many problems, generic goal, successor function & heuristics
  - Just represent problem as a CSP & solve with general package
- CSP “knows” which variables violate a constraint
  - And hence where to focus the search
- *CSPs:* Automatically prune off all branches that violate constraints
  - (State space search could do this only by *hand-building constraints into the successor function)*

CSP Representations

- *Constraint graph:*
  - *nodes* **are variables**
  - *arcs* **are (binary) constraints**
- *Standard representation pattern:*
  - **variables with values**
- *Constraint graph* **simplifies search.**
  - **e.g. Tasmania is an independent subproblem.**
- *This problem: A binary CSP:*
  - **each constraint relates two variables**





Varieties of CSPs

- *Discrete variables*
  - finite domains:
    - n variables, domain size d → $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- *Continuous variables*
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

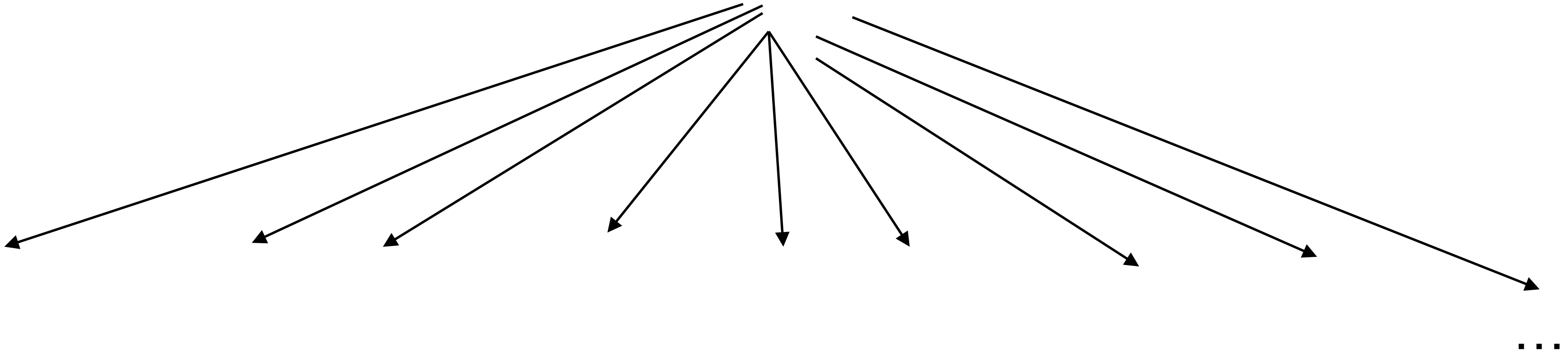
- *Unary* constraints involve a single variable,
  - e.g., SA ≠ green
- *Binary* constraints involve pairs of variables,
  - e.g., SA ≠ WA
- *Higher-order* constraints involve 3 or more variables
  - e.g., crypt-arithmetic column constraints
- *Preference* (soft constraints) e.g. *red* is better than *green* can be represented by a cost for each variable assignment
  - Constrained optimization problems.

Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**
  - *Initial State:* the empty assignment {}.
  - *Successor function:* Assign value to any unassigned variable *provided that there is not a constraint conflict*.
  - *Goal test:* the current assignment is complete.
  - *Path cost:* a constant cost for every step.

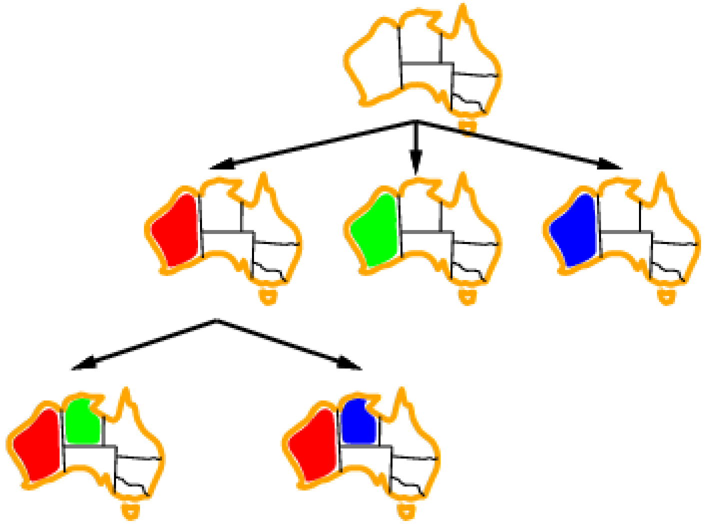
- **Solution is always found at depth *n*, for *n* variables**
  - Hence Depth First Search can be used

Search and branching factor



- n variables of domain size d
- Branching factor at the root is $n*d$
- Branching factor at next level is $(n-1)*d$
- Tree has $n!*d^n$ leaves

Search and branching factor



- The variable assignments are *commutative*
  - Eg *[step 1: **WA = red**; step 2: **NT = green**]* equivalent to *[step 1: **NT = green**; step 2: **WA = red**]*
  - Therefore, a *tree search,* not a *graph search*

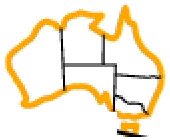
- **Only need to consider assignments to a single variable at each node**
  - $b = d$ and there are d^n leaves *(n variables, domain size d)*

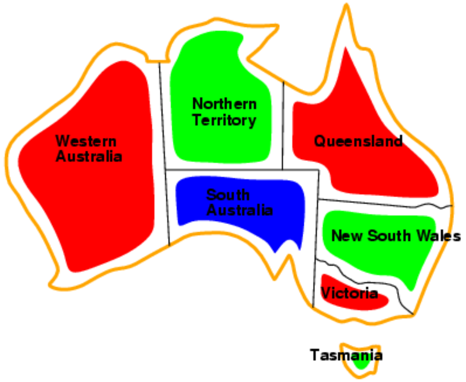
Search and *Backtracking*

- **Depth-first search for CSPs with single-variable assignments is called *backtracking* search**
- **The term backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.**
- **Backtracking search is the basic *uninformed* algorithm for CSPs**

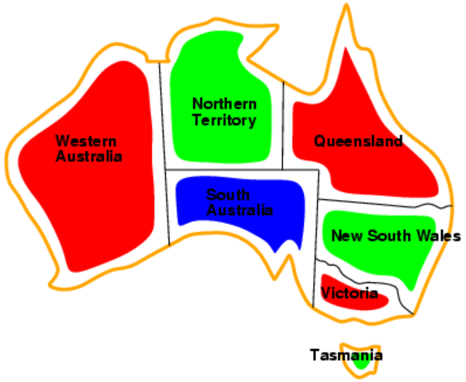
How does this backtracking search differ from our previous formulation of a DFS?

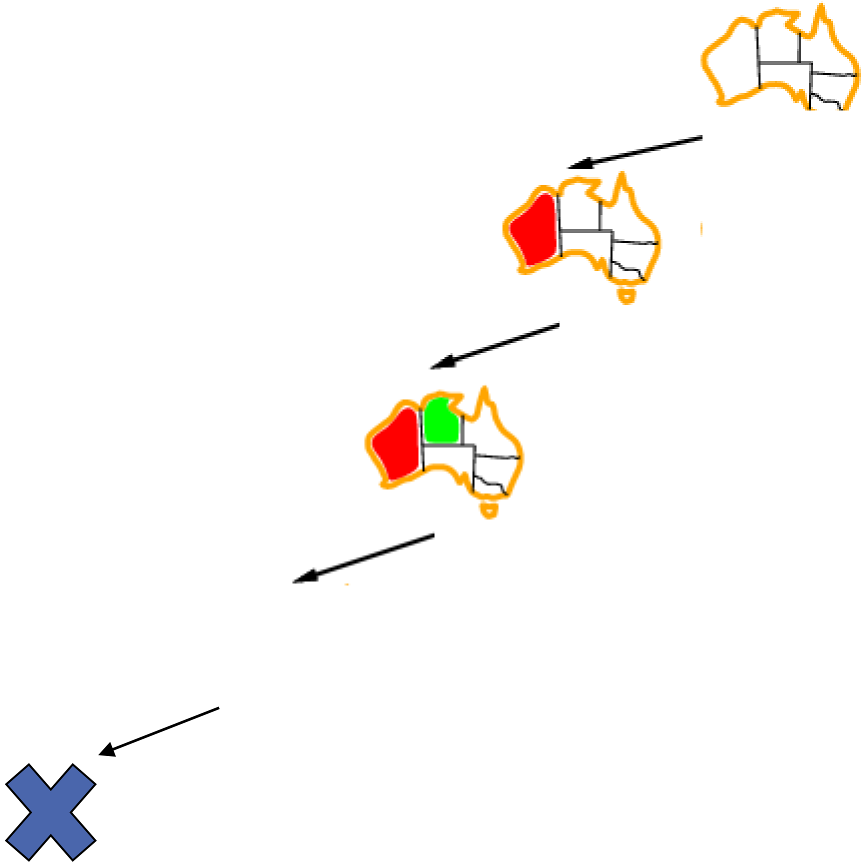
Backtracking example





Backtracking example



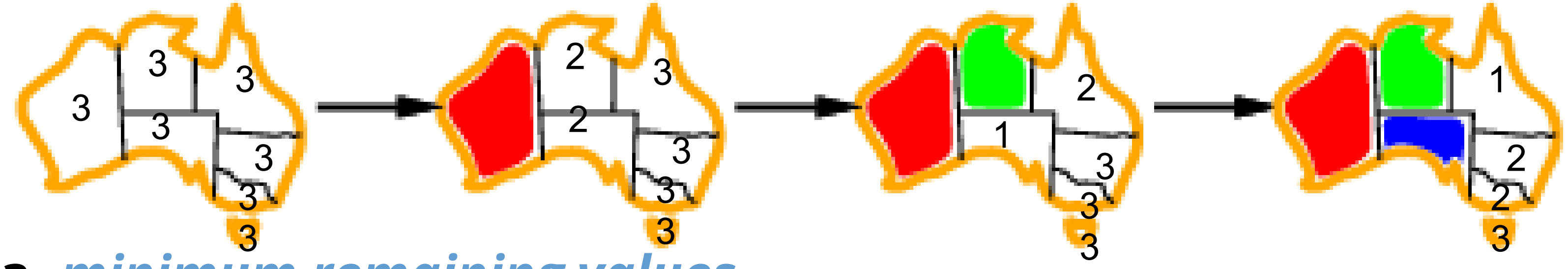


Idea 2: Improving backtracking efficiency

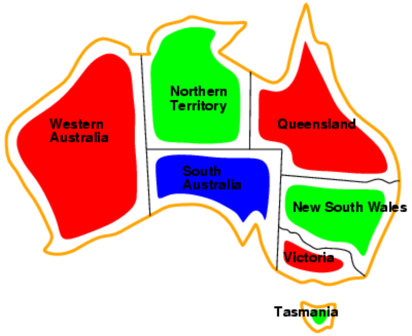
- *General-purpose* methods & *general-purpose* heuristics can give huge gains in speed, *on average*
- Heuristics:
  - Q: Which variable should be assigned next?
    1. **Most constrain*ed* variable**
    2. **(if ties:) Most constrain*ing* variable**
  - Q: In what order should that variable's values be tried?
    3. **Least constraining *value***
  - Q: Can we detect inevitable failure early?
    4. **Forward checking**

Heuristic 1: Most constrain*ed* variable

- **Choose a variable with the *fewest legal values***

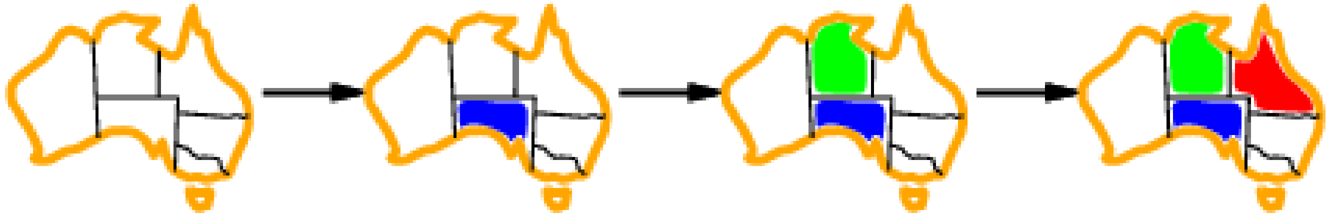


- **a.k.a. *minimum remaining values (MRV)* heuristic**

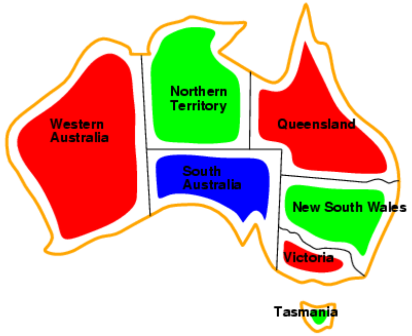


Heuristic 2: Most constrain*ing* variable

- **Tie-breaker among most constrained variables**
- **Choose the variable with the *most constraints on remaining variables***



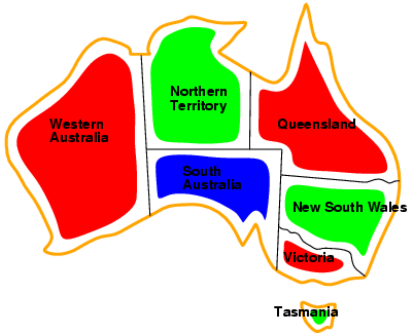
These two heuristics together lead to immediate solution of our example problem



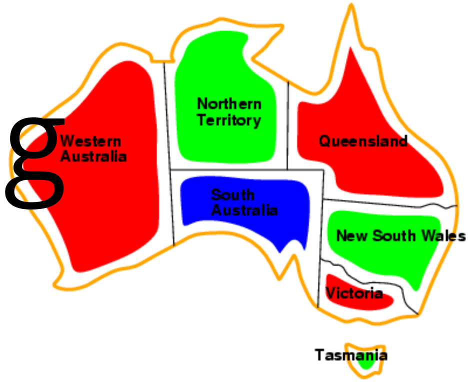
Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  - the one that rules out the fewest values in the remain

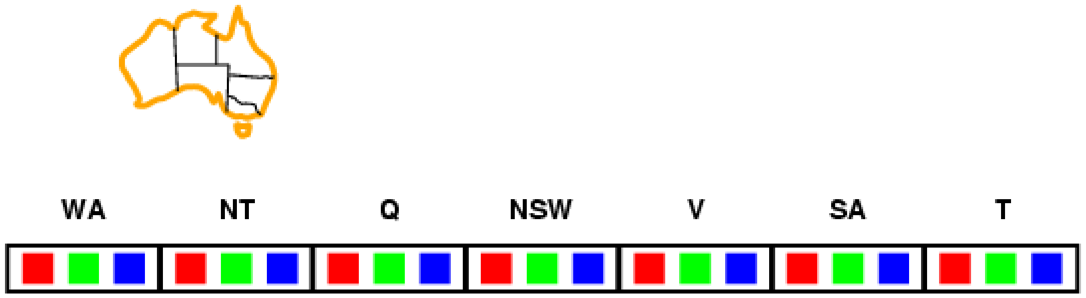
Note: demonstrated here independent of the other heuristics



Heuristic 4: Forward checking

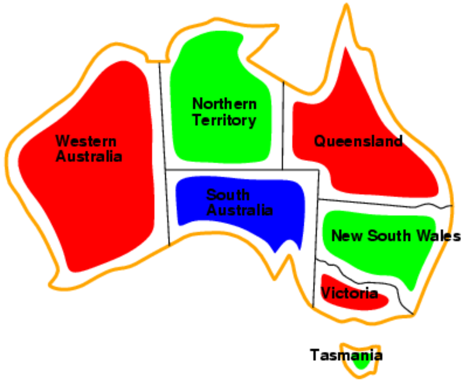


- Idea:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values



New data structure

Forward checking



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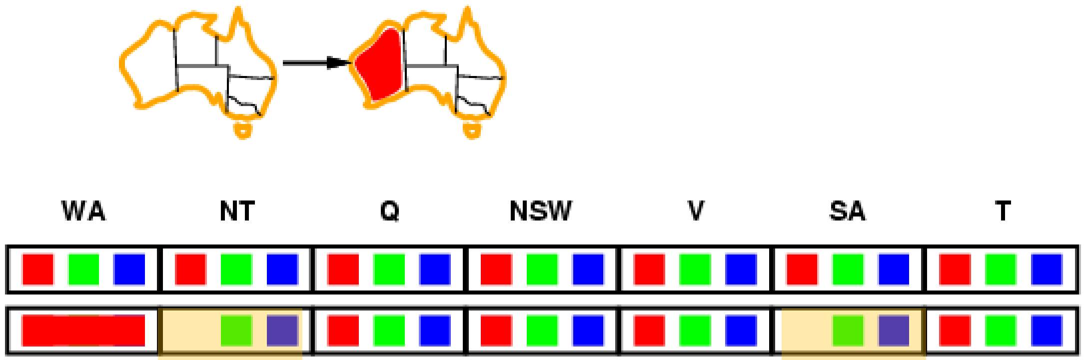
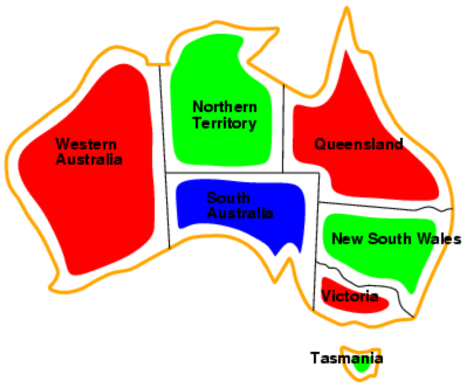


Forward checking



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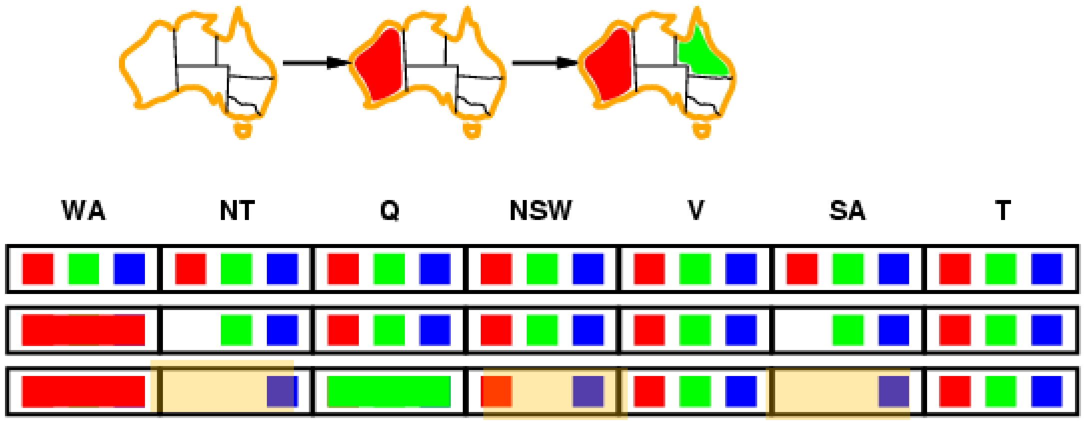
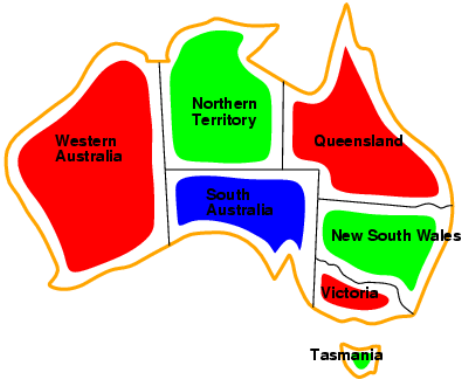


Forward checking



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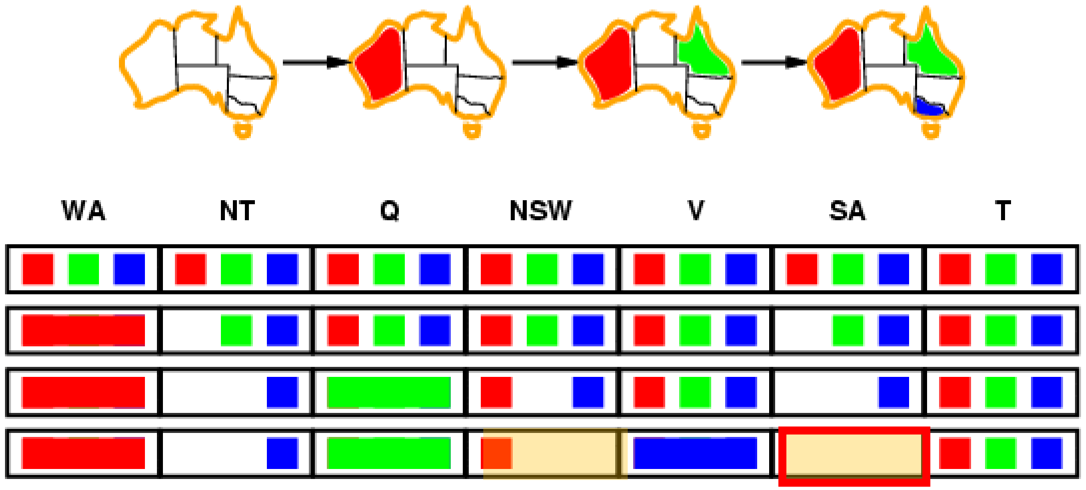
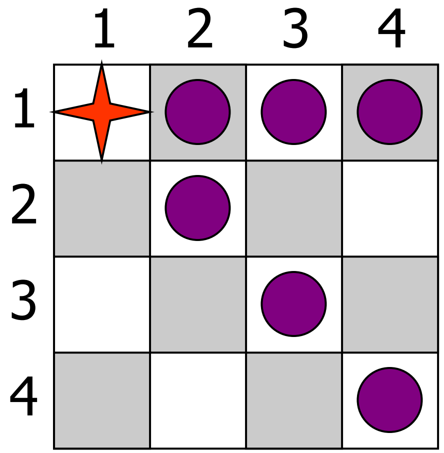
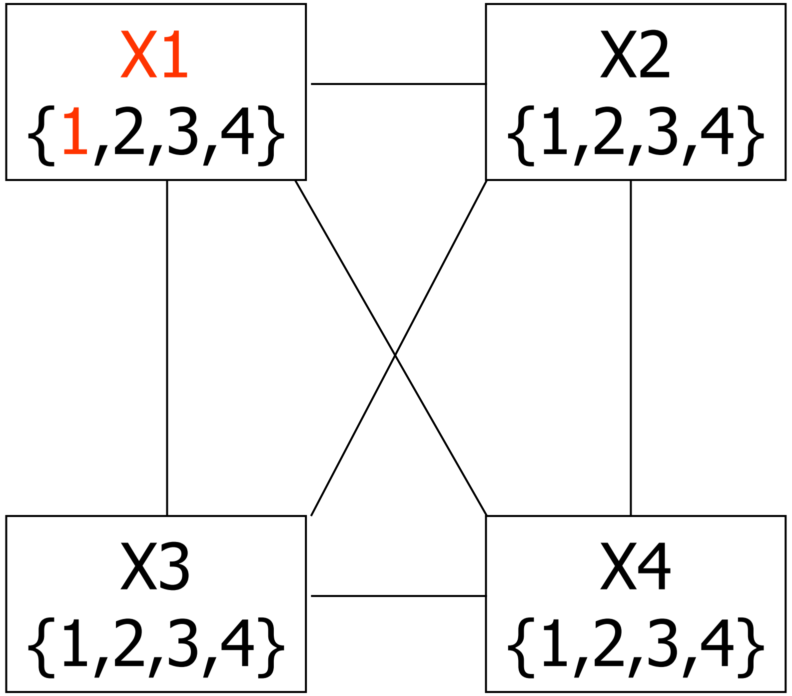


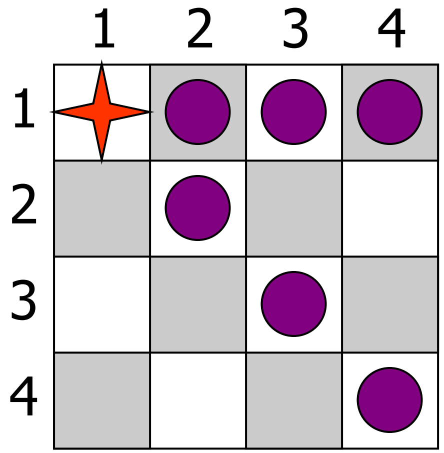
Example: 4-Queens Problem

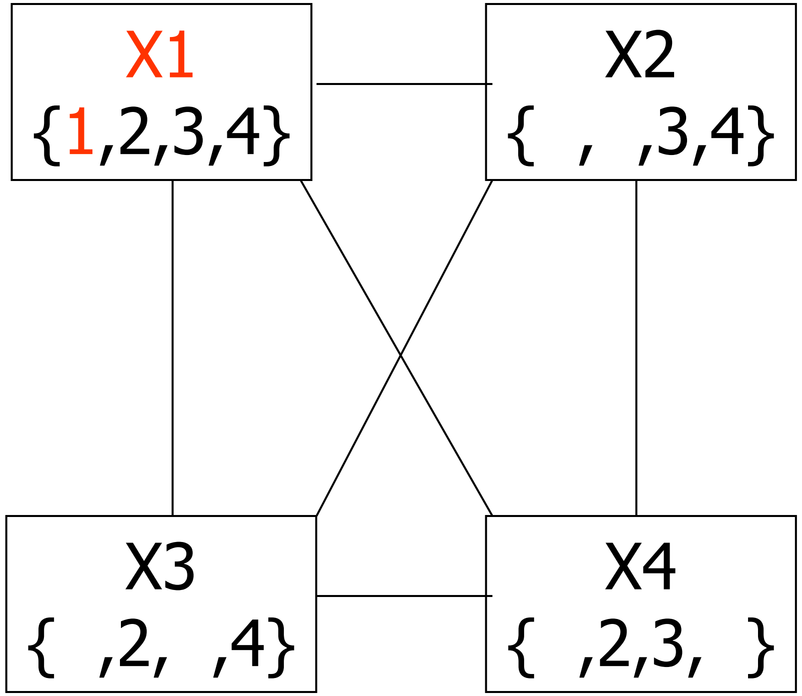




Assign value to unassigned variable

Example: 4-Queens Problem





Forward check!

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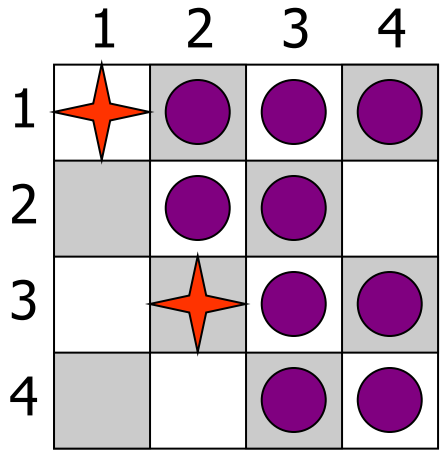

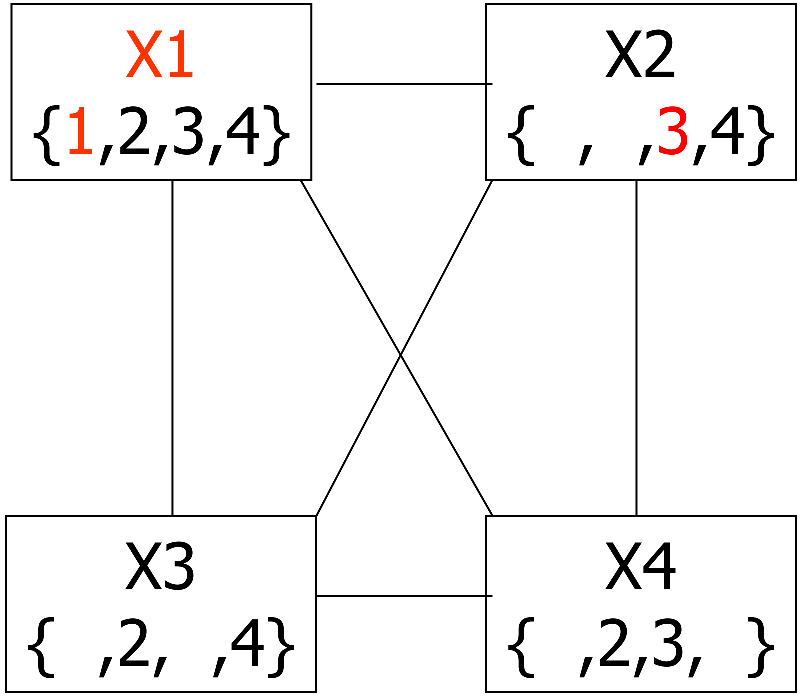

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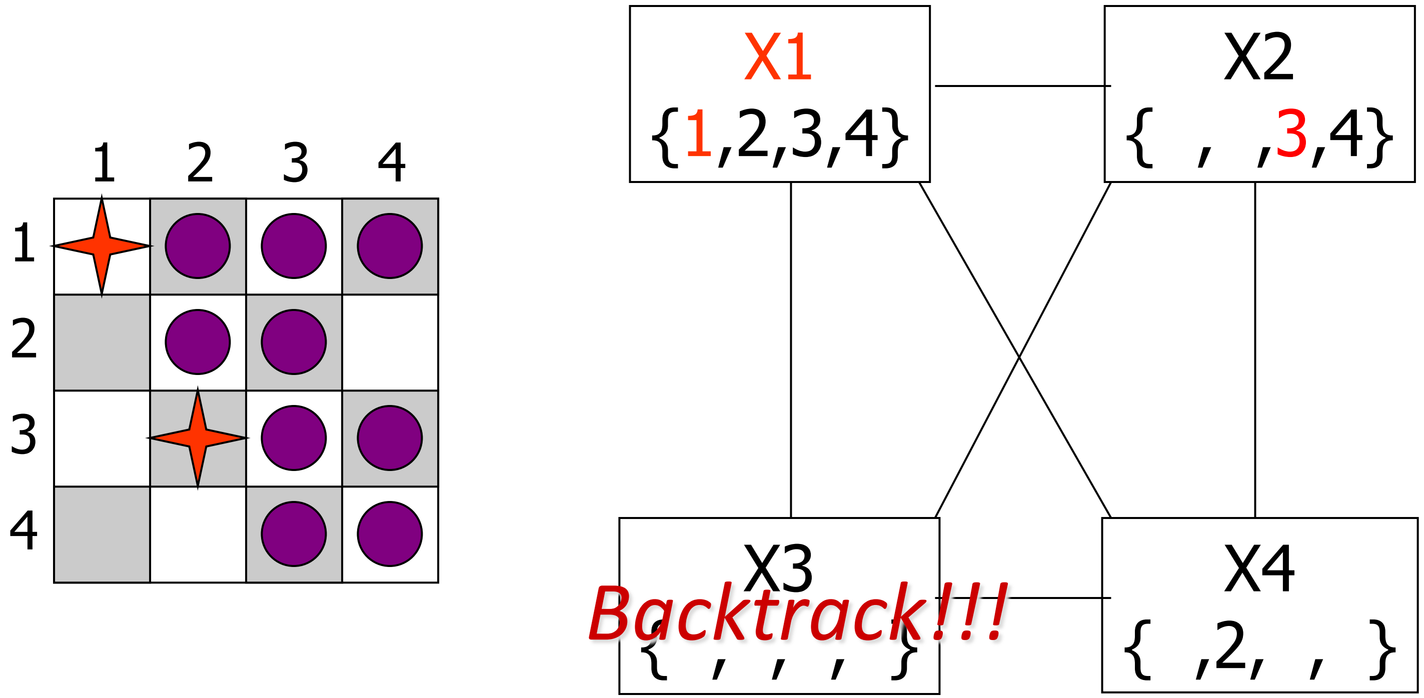
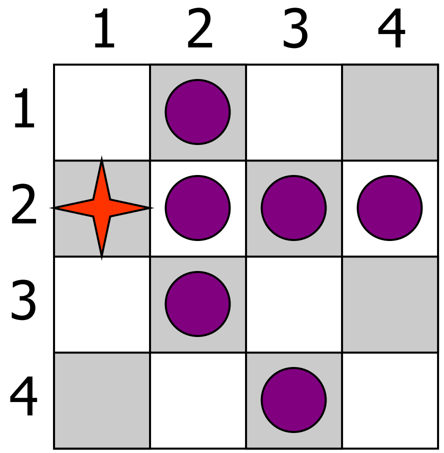
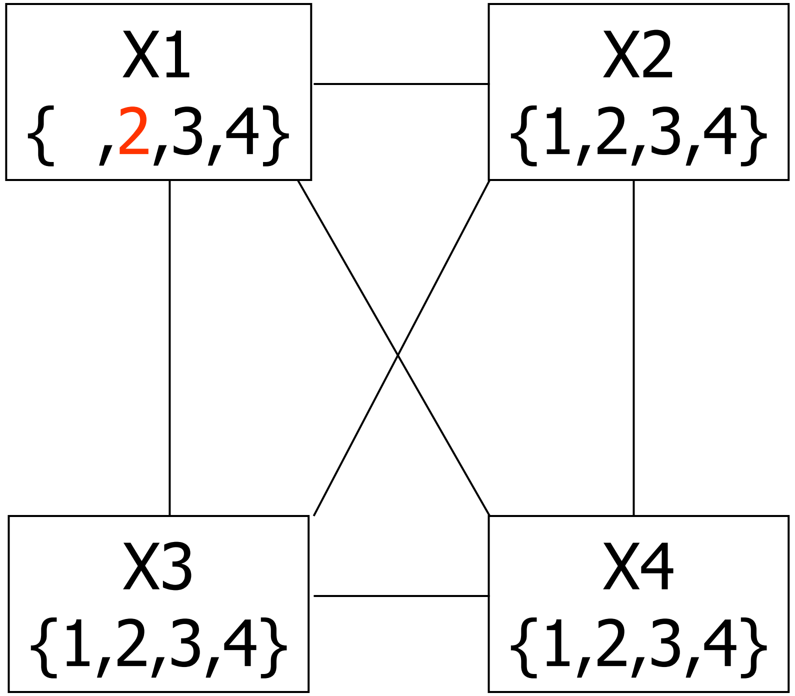


Forward check!

Example: 4-Queens Problem

Picking up a little later after two steps of backtracking....





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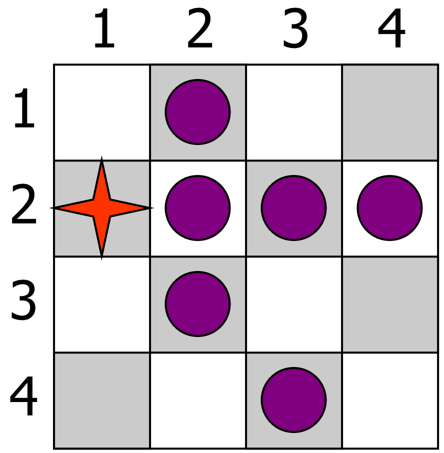

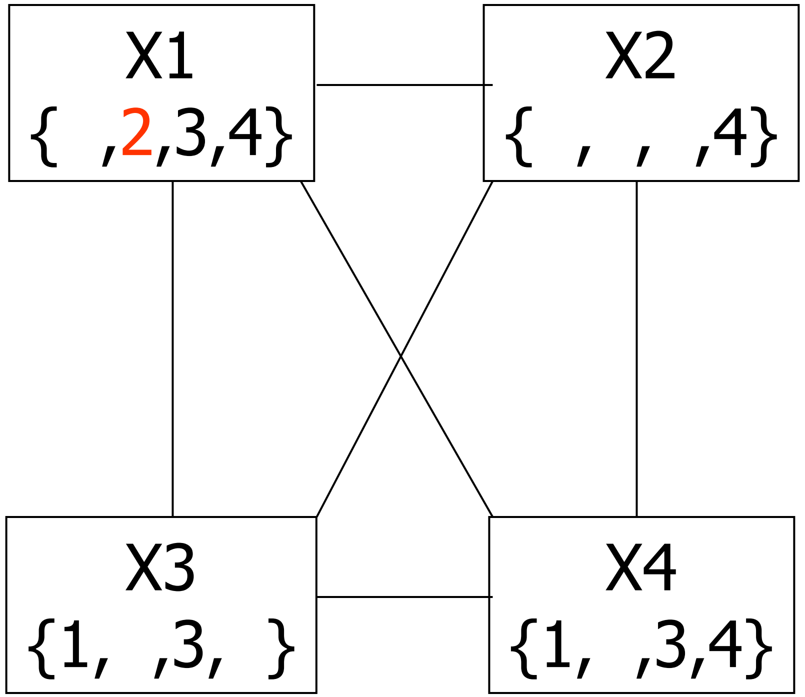

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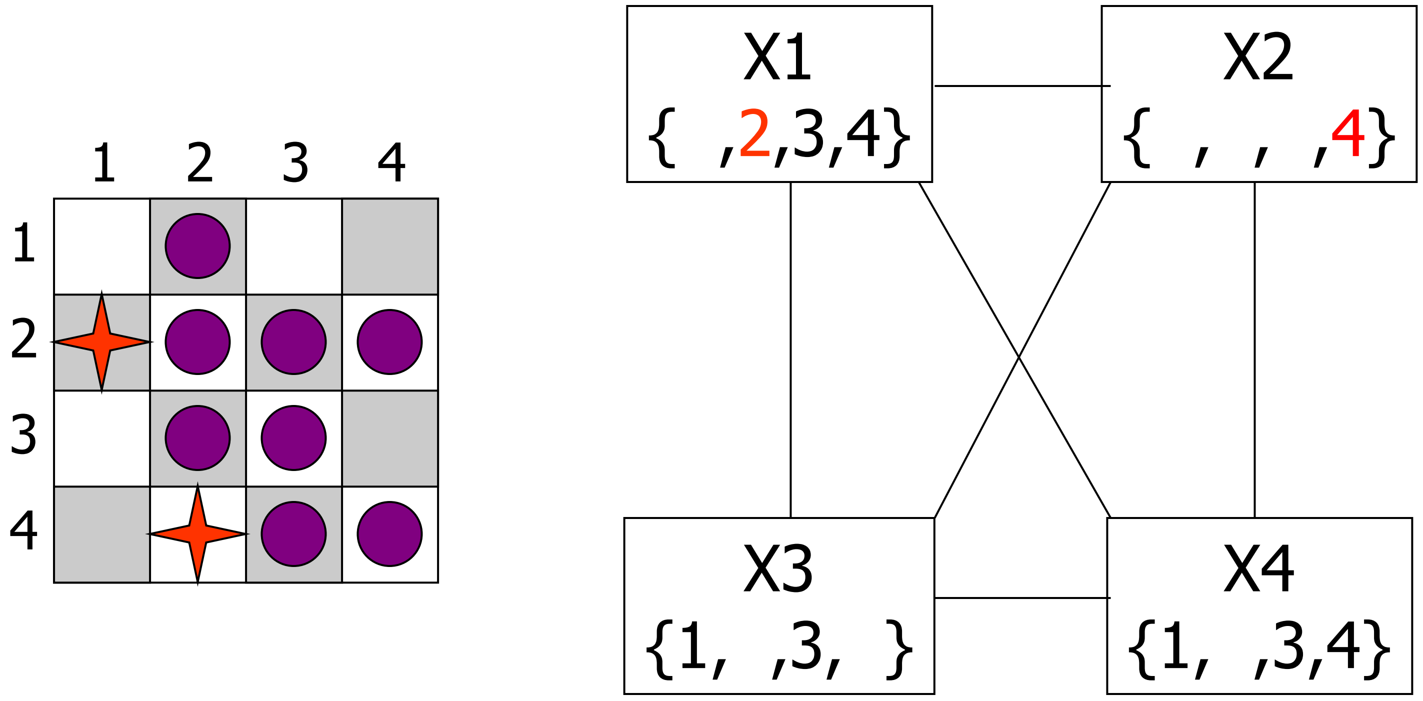

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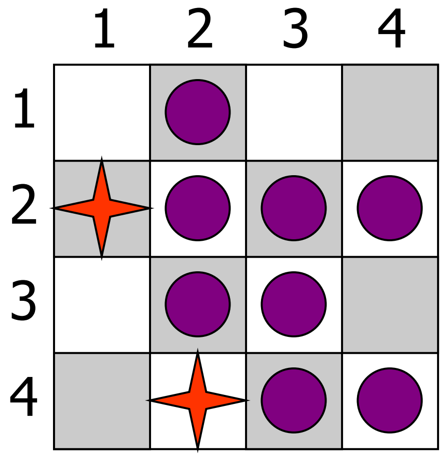

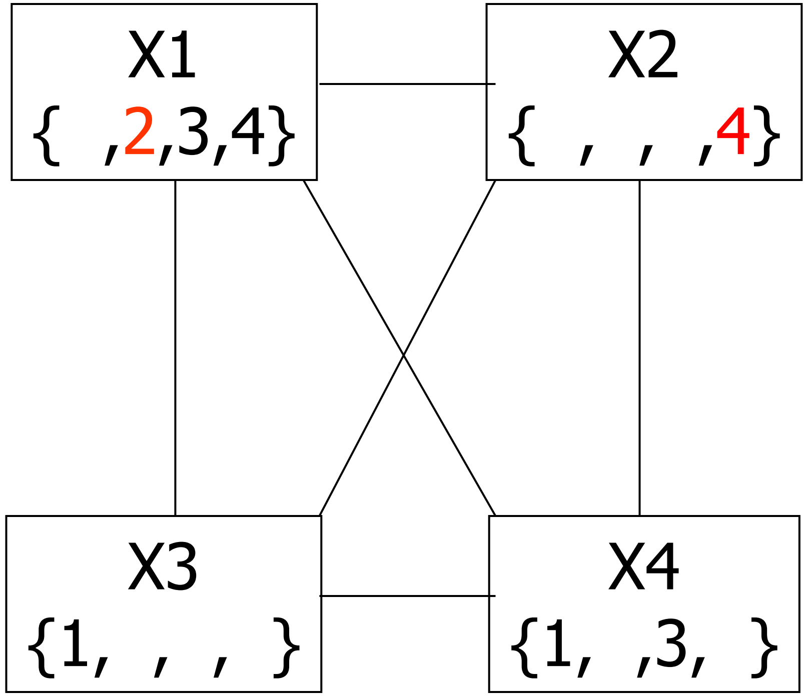

Forward check!

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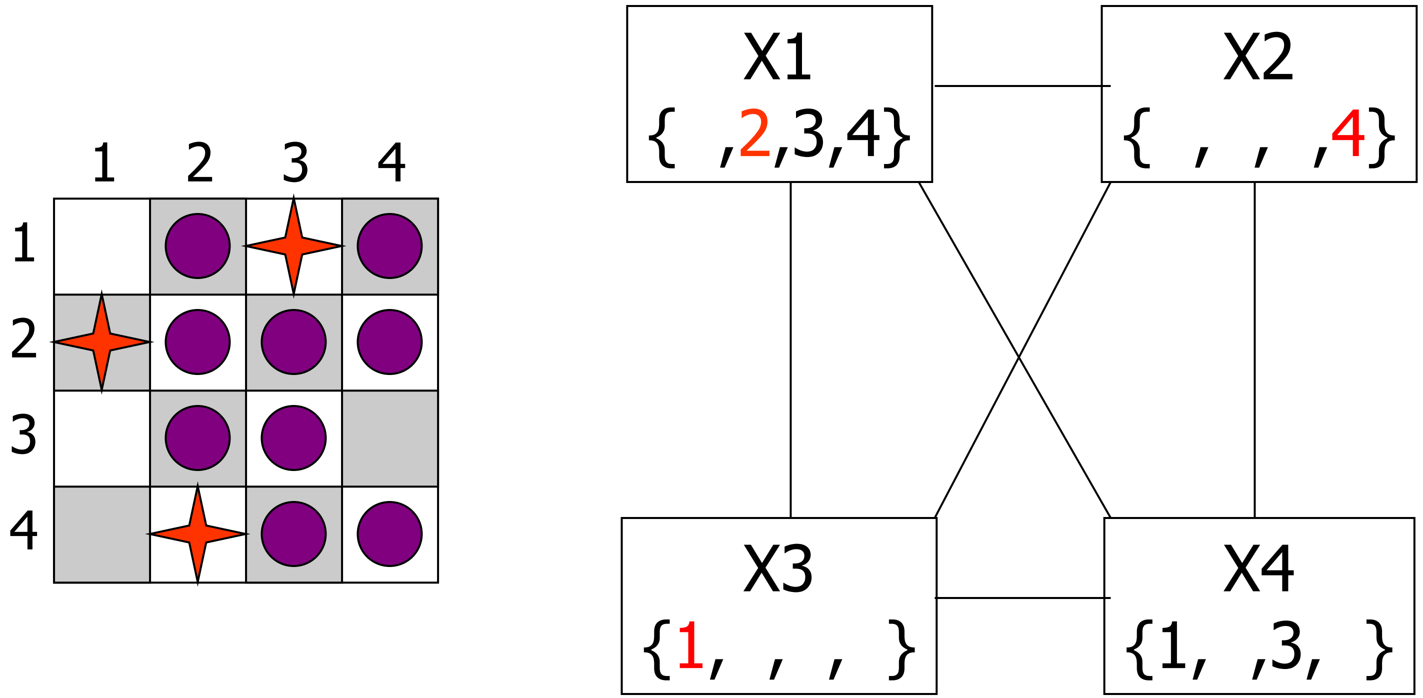

Assign value to unassigned variable

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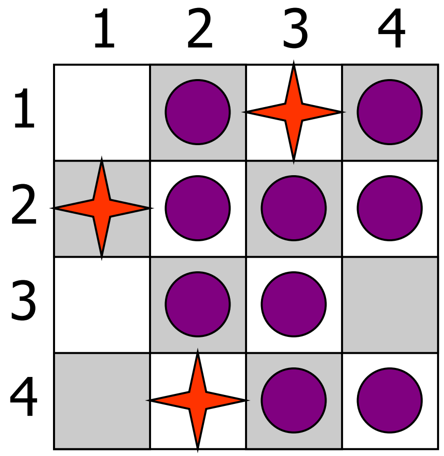

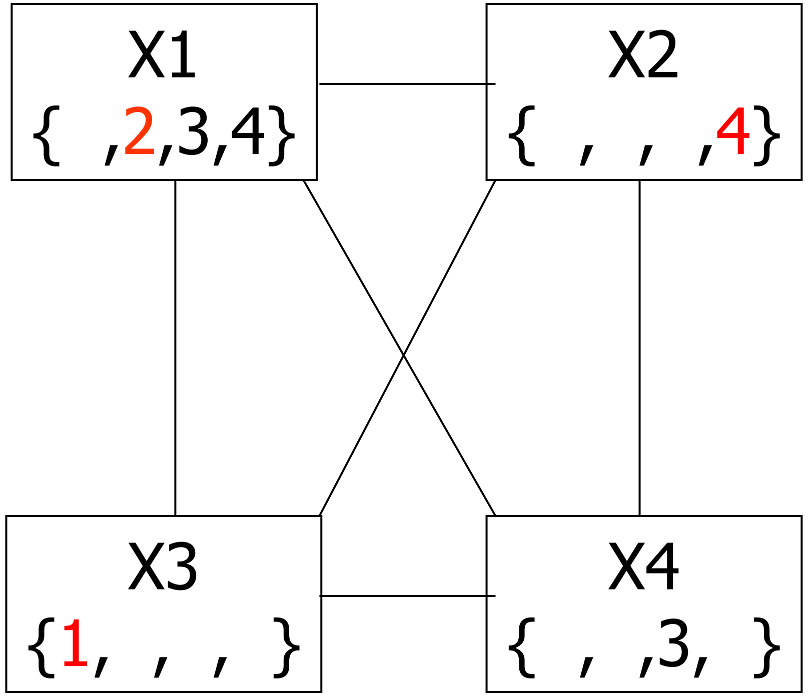

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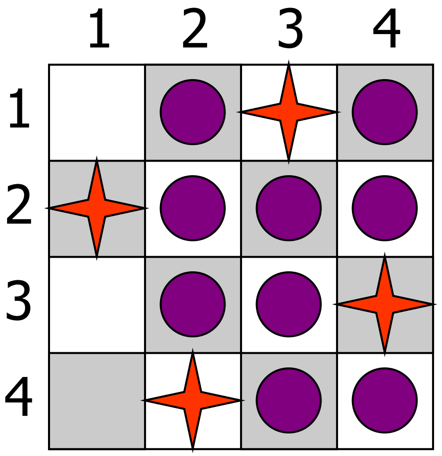
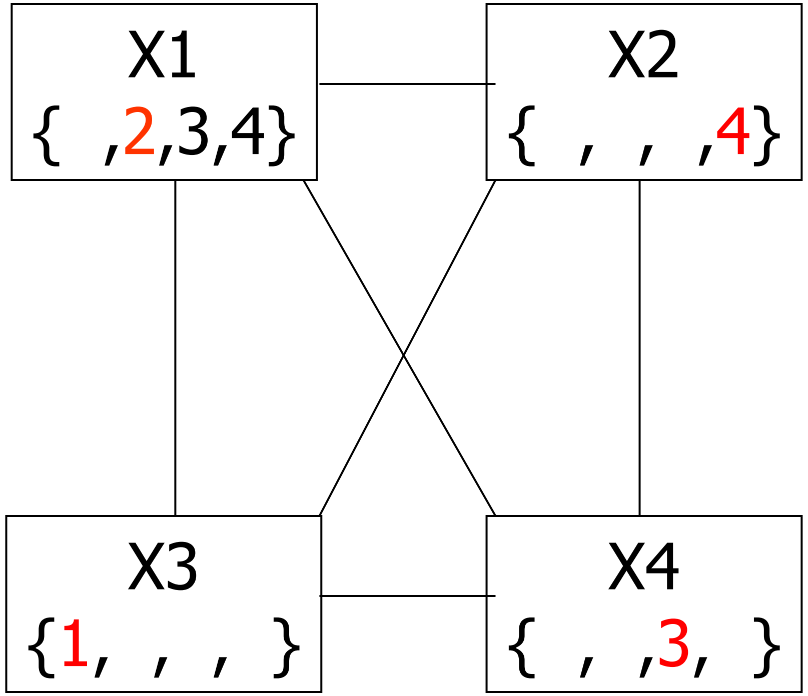
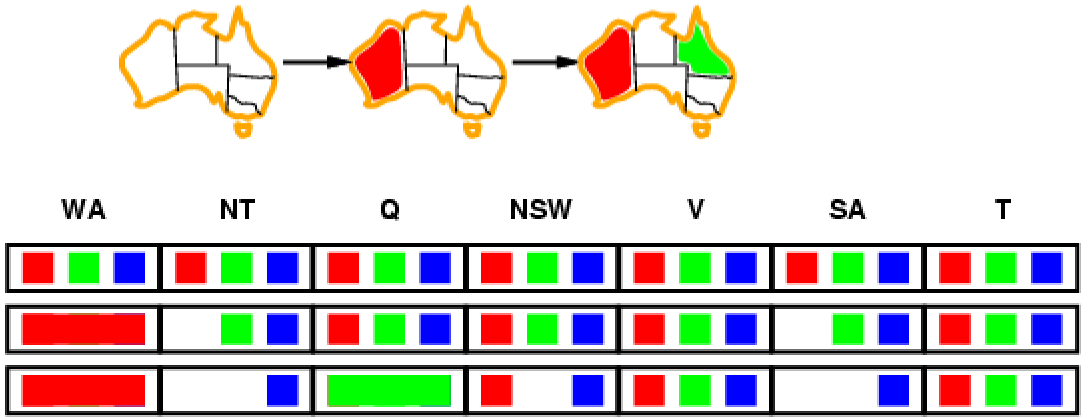




Assign value to unassigned variable

Towards Constraint propagation

- Forward checking propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation goes beyond forward checking & repeatedly enforces constraints locally

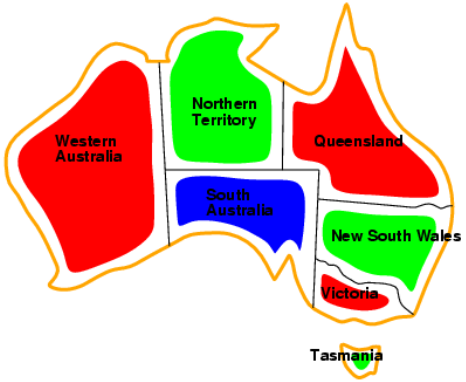
Arc Consistency,
Constraint Propagation & AC-3

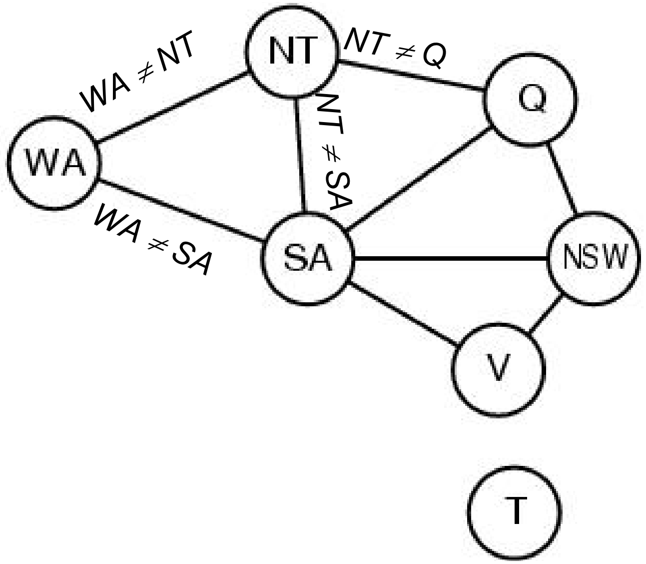
Idea 3 (*big* idea): *Inference* in CSPs

- **CSP solvers combine search *and inference***
  - Search
    - assigning a value to a variable
  - *Constraint propagation (inference)*
    - Eliminates possible values for a variable if the value would violate local consistency
  - *Can do inference first, or intertwine it with search*
    - You'll investigate this in the Sudoku homework
- **Local consistency**
  - Node consistency: satisfies unary constraints
    - This is trivial!
  - *Arc consistency*: satisfies binary constraints
    - (X_i is arc-consistent w.r.t. X_j if for every value v in D_i, there is some value w in D_j that satisfies the binary constraint on the arc between X_i *and* X_j)

CSP Representations

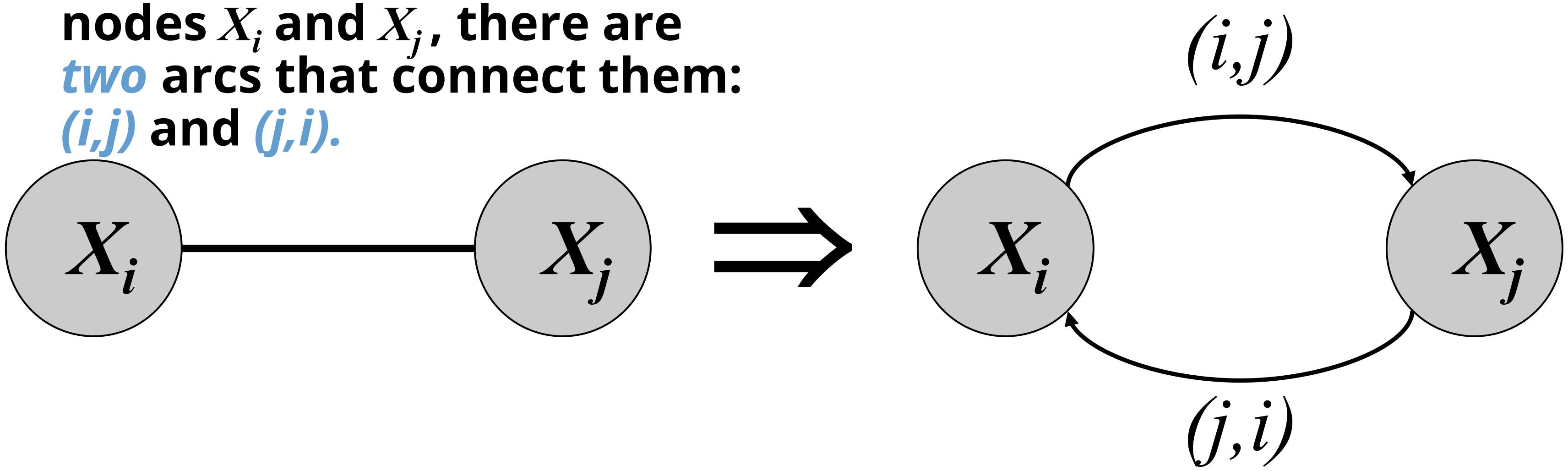
- ***Constraint graph:***
  - *nodes* are variables
  - *edges are constraints*





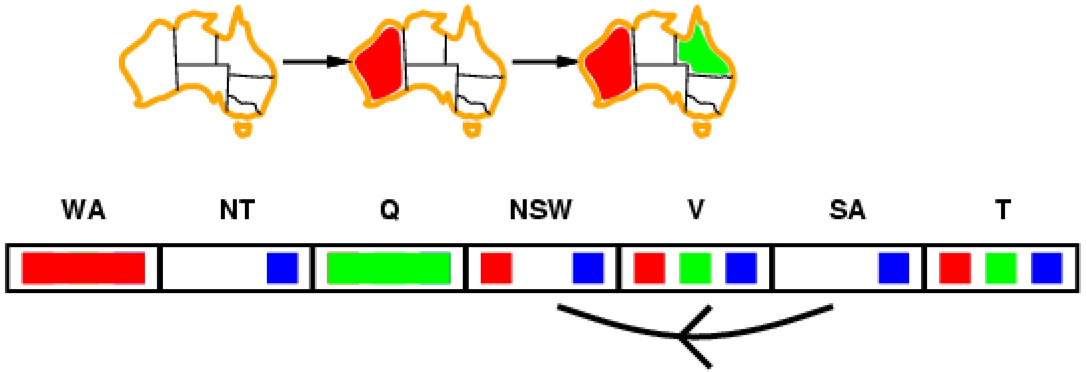
Edges to Arcs: From Constraint Graph to Directed Graph

- **Given a pair of nodes X_i and X_j connected by a constraint *edge*, we represent this not by a single undirected edge, but a *pair of directed arcs*.**
  - **For a connected pair of nodes X_i and X_j, there are *two* arcs that connect them: *(i,j)* and *(j,i)*.**



Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



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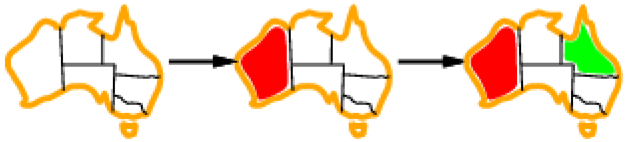

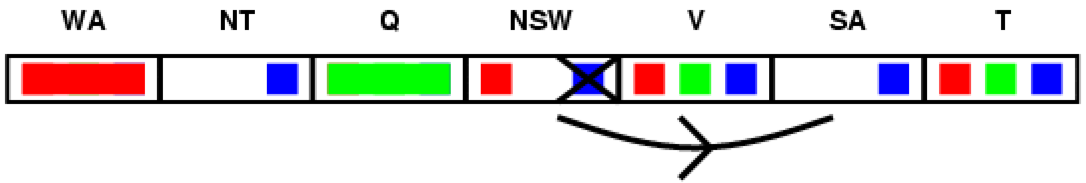

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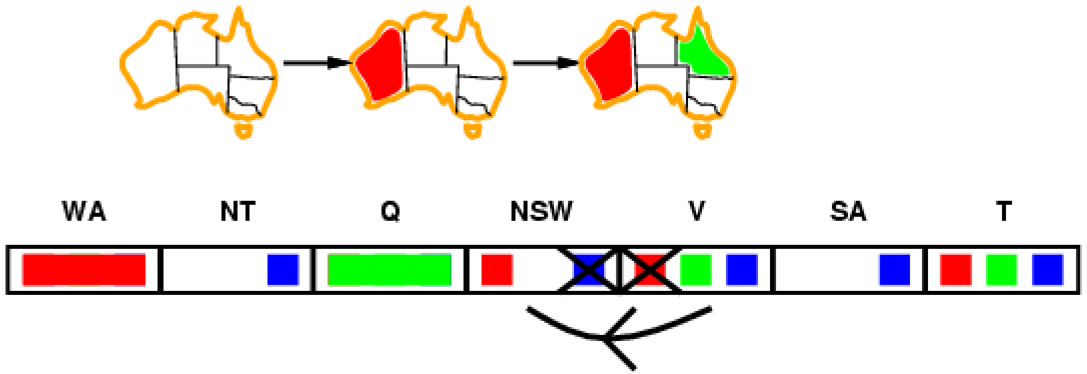

- If X loses a value, recheck neighbors of X

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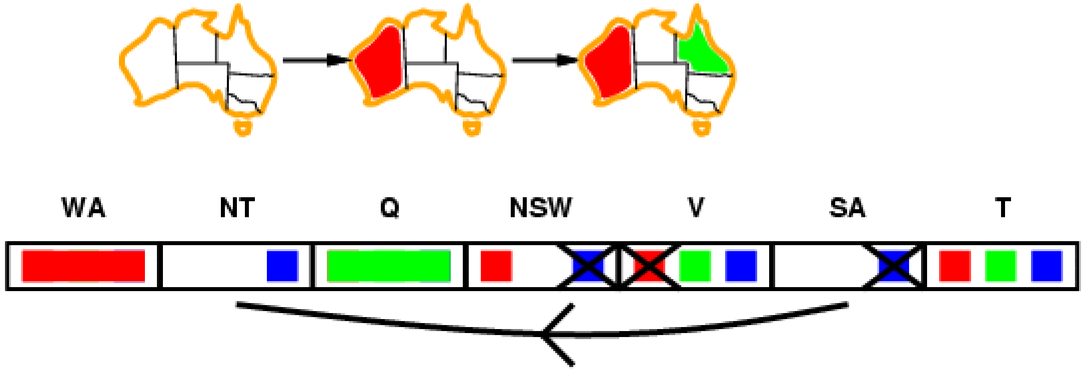


- If X loses a value, we need to recheck neighbors of X
- Detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency

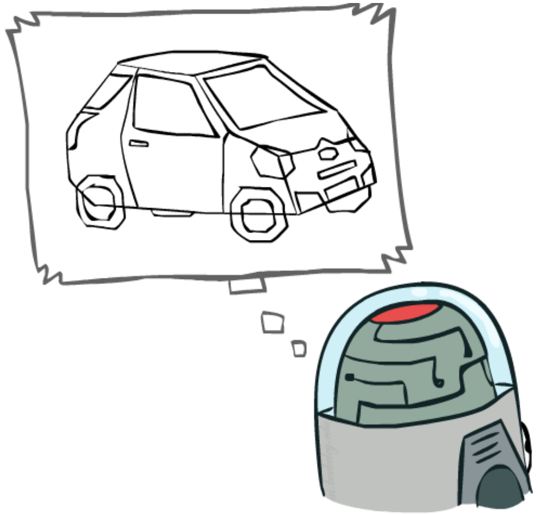
An arc *(i,j) is* arc consistent if and only if every value *v* on X_i is consistent with some label on Y_j.

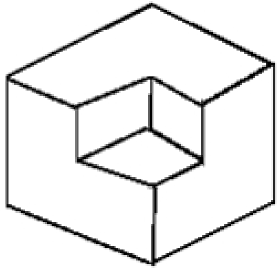
To make an arc *(i,j)* arc consistent,
for each value *v* on X_i ,
if there is no label on Y_j consistent with *v*
then remove *v* from X_i

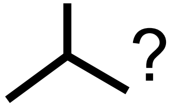
- **Given *d* values, checking arc (i,j) takes *$O(d^2)$* time worst case**

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP





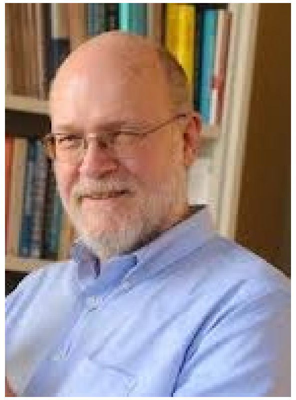


- **Approach:**
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

Replacing Search: Constraint Propagation Invented…

Dave Waltz's insight:



- **By *iterating* over the graph, the arc-consistency *constraints* can be *propagated* along arcs of the graph.**
- ***Search*: Use constraints to *add* labels to find *one solution***
- ***Constraint Propagation*: Use constraints to *eliminate labels* to simultaneously find *all solutions***

The Waltz/Mackworth Constraint Propagation Algorithm

1. **Assign *every* node in the constraint graph a set of *all* possible values**
2. **Repeat until there is no change in the set of values associated with any node:**
   3. For each node i:
      4. For each neighboring node j in the picture:
         5. Remove any value from i which is not arc consistent with j.

Inefficiencies: Towards AC-3

1. **At each iteration, we only need to examine those X_i *where at least one neighbor of X_i has lost a value* in the previous iteration.**
2. **If X_i loses a value only because of arc inconsistencies with Y_j, we *don't need to check* Y_j on the next iteration.**
3. **Removing a value on X_i can only make Y_j arc-inconsistent with respect to X_i itself. Thus, we only need to check that *(j,i)* is still arc-consistent.**

These insights lead a much better algorithm…

AC-3

function AC-3(*csp*) return the CSP, possibly with reduced domains

inputs: *csp*, a binary csp with variables $\{X_1, X_2, \ldots, X_n\}$

local variables: *queue,* a queue of arcs initially the arcs in *csp*

while queue is not empty do

$(X_i, X_j) \leftarrow$ ***queue.***pop()

if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then

for each X_k in NEIGHBORS[X_i] – {X_j} do
add (X_k, X_i) to queue

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) return *true* iff we remove a value

removed $\leftarrow$ ***false***

for each *x* in DOMAIN[X_i] do

if no value *y* in DOMAIN[X_j] allows (x,y) to satisfy
the constraints between X_i and X_j

then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ ***true***

return *removed*

Keep track of what arcs we need to process

Add back arcs to neighbors whenever a node had values removed

AC-3: Worst Case Complexity Analysis

- **All nodes can be connected to *every* other node,**
  - so each of ***n*** nodes must be compared against ***n-1*** other nodes,
  - so total # of arcs is ***2*n*(n-1), i.e. O(n^2)***
- **If there are *d* values, checking arc (i,j) takes *O(d^2)* time**
- **Each arc (i,j) can only be inserted into the queue *d* times**
- **Worst case complexity: *O(n^2d^3)***

(For *planar* constraint graphs, the number of arcs can only be *linear in N and* the time complexity is only *O(nd^3)*)